

A MODULAR FRAMEWORK FOR FUNCTIONS OF KNOPP AND INDEFINITE BINARY QUADRATIC FORMS

KATHRIN BRINGMANN AND ANDREAS MONO

ABSTRACT. We study functions introduced by Knopp and complete them to non-holomorphic bimodular forms of positive integral weight related to indefinite binary quadratic forms. We investigate further properties of our completions, which in turn motivates certain local cusp forms. We then define modular analogues of negative weight of our local cusp forms, which are locally harmonic Maass forms with continuously removable singularities. We show that they admit local splittings in terms of Eichler integrals.

1. INTRODUCTION AND STATEMENT OF RESULTS

Throughout the paper $D > 0$ is a non-square discriminant, $k \in 2\mathbb{N}$, \mathcal{Q}_d denotes the set of integral binary quadratic forms $Q = [a, b, c]$ of discriminant $d \in \mathbb{Z}$, and \mathbb{H} is the complex upper half-plane. In 1975, Zagier [29] introduced the functions¹

$$f_{\kappa,D}(\tau) := \sum_{Q \in \mathcal{Q}_D} \frac{1}{Q(\tau, 1)^\kappa}, \quad \tau \in \mathbb{H},$$

and proved that they are weight 2κ cusp forms if $\kappa > 1$ (if $\kappa = 1$, one may use Hecke's trick, see [19, p. 239]). To name a few prominent applications of the $f_{\kappa,D}$, they are coefficients of the holomorphic kernel function of the Shimura [26] and Shintani [27] lifts due to [19], and they are closely related to central L -values by [20]. Their even periods are rational according to [21], and they generate the space of weight 2κ cusp forms [12].

Over 30 years ago, Knopp [13, (4.5)] found a term-by-term preimage of each $f_{\kappa,D}$ under the *Bol operator* \mathbb{D}^{2k-1} , where $\mathbb{D} := \frac{1}{2\pi i} \frac{\partial}{\partial \tau}$ (compare Proposition 3.1 (2)). We refer to Knopp's earlier work [14–16] and to the references [6, 9, 22] for the importance of the Bol operator. To ensure convergence after summing over $Q \in \mathcal{Q}_D$, Knopp changed the sign of k in his result afterwards, which lead to (throughout Log denotes the principal branch of the complex logarithm)

$$\psi_{k+1,D}(\tau) := \sum_{Q \in \mathcal{Q}_D} \frac{\text{Log} \left(\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+} \right)}{Q(\tau, 1)^{k+1}}, \quad \alpha_{[a,b,c]}^\pm := \frac{-b \pm \sqrt{D}}{2a} \in \mathbb{R}. \quad (1.1)$$

He also stated that $\psi_{k+1,D}(\tau + 1) = \psi_{k+1,D}(\tau)$, and the behaviour of $\psi_{k+1,D}$ under modular inversion² (see [13, (4.6)]). Correcting a typo there, we find that (see Proposition 3.1 (3))

$$\tau^{-2k-2} \psi_{k+1,D} \left(-\frac{1}{\tau} \right) - \psi_{k+1,D}(\tau) = \sum_{Q \in \mathcal{Q}_D} \frac{\log \left| \frac{\alpha_Q^+}{\alpha_Q^-} \right|}{Q(\tau, 1)^{k+1}} - 2\pi i \sum_{\substack{Q=[a,b,c] \in \mathcal{Q}_D \\ a < 0 < c}} \frac{1}{Q(\tau, 1)^{k+1}}. \quad (1.2)$$

On the one hand, we observe that $\psi_{k+1,D}$ is holomorphic and vanishes at $i\infty$ (this follows by Proposition 3.1 (1) and (3.9)). On the other hand, $\psi_{k+1,D}$ itself is not modular. Hence,

Date: September 12, 2025.

2020 Mathematics Subject Classification. 11F11 (primary); 11E16, 11E45, 11F37 (secondary).

Key words and phrases. Harmonic Maass forms, integral binary quadratic forms, locally harmonic Maass forms, modular completion, theta lifts.

¹We define $f_{\kappa,D}$ in Zagier's original normalization, which differs from the normalization used in [3].

²We alert the reader to the fact that Knopp used the older convention $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

it is natural to “complete” $\psi_{k+1,D}$. Setting $\mathbb{H}^- := -\mathbb{H}$ throughout, completions of $\psi_{k+1,D}$ are bimodular forms³ $\Omega_{k+1,D}$ of weight $(2k+2, 0)$ defined on $\mathbb{H} \times \mathbb{H}^-$ such that

$$\lim_{w \rightarrow -i\infty} \Omega_{k+1,D}(\tau, w) = \psi_{k+1,D}(\tau). \quad (1.3)$$

Thus, from the completions $\Omega_{k+1,D}$ one can uniquely recover the original functions $\psi_{k+1,D}$.

In this paper, we construct such completions explicitly. Firstly, we note that the final sum appearing in (1.2) is finite, because $b^2 + 4|ac| = D > 0$ has only finitely many integral solutions. This leads to Knopp’s modular integrals with rational period functions [17, 18]. Roughly speaking, period polynomials describe the obstruction of modularity of Eichler integrals [9] (defined in (1.8)) of cusp forms, and Knopp generalized this notion to rational functions instead of polynomials. Such functions are called *modular integrals*. Parson [25] constructed such modular integrals explicitly by letting

$$\varphi_{k+1,D}(\tau) := \frac{1}{2} \sum_{Q \in \mathcal{Q}_D} \frac{\text{sgn}(Q)}{Q(\tau, 1)^{k+1}} = \sum_{\substack{Q=[a,b,c] \in \mathcal{Q}_D \\ a > 0}} \frac{1}{Q(\tau, 1)^{k+1}}, \quad \text{sgn}([a, b, c]) := \text{sgn}(a) \quad (1.4)$$

and we recall her result on the $\varphi_{k+1,D}$ in Lemma 3.3. Secondly, we define

$$Q_w := \frac{1}{\text{Im}(w)} (a|w|^2 + b \text{Re}(w) + c), \quad S_Q := \{\tau \in \mathbb{H} : Q_\tau = 0\}, \quad E_D := \bigcup_{Q \in \mathcal{Q}_D} S_Q,$$

for $w \in \mathbb{C} \setminus \mathbb{R}$, $Q \in \mathcal{Q}_d$ ($d \in \mathbb{Z}$), as well as the functions

$$\rho_{k+1,D}(\tau, w) := \sum_{Q \in \mathcal{Q}_D} \frac{\text{Log}\left(\frac{w - \alpha_Q^-}{w - \alpha_Q^+}\right)}{Q(\tau, 1)^{k+1}}, \quad \lambda_{k+1,D}(\tau, w) := 2i \sum_{Q \in \mathcal{Q}_D} \frac{\arctan\left(\frac{Q_w}{\sqrt{D}}\right)}{Q(\tau, 1)^{k+1}}. \quad (1.5)$$

for $w \in \mathbb{H}^-$. We refer to Propositions 3.2 and 3.4 for some of their properties.

Thirdly, we define completions of $\psi_{k+1,D}$ as⁴

$$\Omega_{k+1,D}(\tau, w) := \psi_{k+1,D}(\tau) - \rho_{k+1,D}(\tau, w) + 2\pi i \varphi_{k+1,D}(\tau) + \lambda_{k+1,D}(\tau, w), \quad (1.6)$$

on $\mathbb{H} \times \mathbb{H}^-$. In Proposition 3.4 (2), we show that the functions $\lambda_{k+1,D}$ are themselves bimodular of the same weights as the $\Omega_{k+1,D}$. Their purpose is to ensure parts (2) to (5) of the following theorem (compare Proposition 3.4 (3) and (3.13)). In particular, the functions $\lambda_{k+1,D}$ ensure that the $\Omega_{k+1,D}$ from (1.6) satisfy (1.3).

Theorem 1.1. *Let $\tau \in \mathbb{H}$, $w \in \mathbb{H}^-$.*

(1) *The functions $\Omega_{k+1,D}$ are bimodular of weight $(2k+2, 0)$ that is*

$$\Omega_{k+1,D}(\tau + 1, w + 1) = \Omega_{k+1,D}(\tau, w), \quad \Omega_{k+1,D}\left(-\frac{1}{\tau}, -\frac{1}{w}\right) = \tau^{2k+2} \Omega_{k+1,D}(\tau, w).$$

(2) *We have*

$$\lim_{w \rightarrow -i\infty} \Omega_{k+1,D}(\tau, w) = \psi_{k+1,D}(\tau).$$

(3) *We have*

$$\lim_{\tau \rightarrow i\infty} \Omega_{k+1,D}(\tau, w) = 0.$$

(4) *The functions $\Omega_{k+1,D}$ are holomorphic with respect to τ and anti-holomorphic with respect to w .*

(5) *We have that*

$$\Omega_{k+1,D}(\tau, \bar{\tau}) = 0.$$

³We slightly modify the initial definition by Stienstra and Zagier [28] to include the domain $\mathbb{H} \times \mathbb{H}^-$.

⁴The name completion is justified by Theorem 1.1 (2).

Remark. In [7, Theorem 3], Duke, Imamoglu, and Tóth constructed (weakly) holomorphic modular integrals F with rational period functions. It turns out that Parson's modular integral $\varphi_{k+1,D}$ has the same rational period functions as the ones arising from F , see Lemma 3.3. In [7, (16)], Duke, Imamoglu, and Tóth showed that both modular integrals differ by a (non-explicit) cusp form of weight greater than 2. For weight 2, the second author [23, Theorem 1.1] showed that both modular integrals differ by an explicit non-zero multiple of the weight 2 non-holomorphic Eisenstein series \widehat{E}_2 . Theorem 1.1 embeds Parson's modular integral $\varphi_{k+1,D}$ into a non-holomorphic bimodular framework in higher weights.

Theorem 1.1 explains how modular integrals may exhibit modular properties by completing them to non-holomorphic bimodular forms. It turns out that the $\Omega_{k+1,D}$ can be extended to $\mathbb{H} \times \mathbb{H}$, and become holomorphic bimodular forms with analogous properties there. Being more precise, we define the functions

$$\omega_{k+1,D}(\tau, z) := \psi_{k+1,D}(\tau) - \sum_{Q \in \mathcal{Q}_D} \frac{\text{Log} \left(\frac{z - \alpha_Q^-}{z - \alpha_Q^+} \right)}{Q(\tau, 1)^{k+1}}, \quad \tau, z \in \mathbb{H},$$

and obtain the following corollary.

Corollary 1.2. *Let $\tau, z \in \mathbb{H}$. We have*

$$\omega_{k+1,D}(\tau, z) = \Omega_{k+1,D}(\tau, \bar{z}).$$

In other words, the $\omega_{k+1,D}$ satisfy

$$\begin{aligned} \omega_{k+1,D}(\tau + 1, z + 1) &= \omega_{k+1,D}(\tau, z), & \omega_{k+1,D}\left(-\frac{1}{\tau}, -\frac{1}{z}\right) &= \tau^{2k+2} \omega_{k+1,D}(\tau, z), \\ \lim_{z \rightarrow i\infty} \omega_{k+1,D}(\tau, z) &= \psi_{k+1,D}(\tau), & \lim_{\tau \rightarrow i\infty} \omega_{k+1,D}(\tau, z) &= 0, \\ \omega_{k+1,D}(\tau, \tau) &= 0, \end{aligned}$$

and are holomorphic with respect to τ and z .

To prove Theorem 1.1 (5), we specialize the $\lambda_{k+1,D}$ to $(\tau, w) = (\tau, \bar{\tau}) \in \mathbb{H} \times \mathbb{H}^-$. Roughly speaking, (3.12) demonstrates that they compensate for the singularities of

$$\Omega_{k+1,D}(\tau, \bar{\tau}) - \lambda_{k+1,D}(\tau, \bar{\tau}) = \sum_{Q \in \mathcal{Q}_D} \frac{\text{Log} \left(\frac{\frac{Q_\tau - i}{\sqrt{D}}}{\frac{Q_\tau + i}{\sqrt{D}}} \right)}{Q(\tau, 1)^{k+1}} + \pi i \sum_{Q \in \mathcal{Q}_D} \frac{\text{sgn}(Q_\tau)}{Q(\tau, 1)^{k+1}}$$

on the set E_D . The functions

$$\Lambda_{k+1,D}(\tau) := \sum_{Q \in \mathcal{Q}_D} \frac{\text{sgn}(Q_\tau)}{Q(\tau, 1)^{k+1}} \tag{1.7}$$

appeared first in a paper [24] by the second author, and turn out to be *local cusp forms*. That is, they behave like cusp forms of weight $2k + 2$ outside E_D , however, in addition, have jumping singularities⁵ on E_D . A full definition of such functions can be found in [24, Definition 2.7], which in turn adapts an earlier definition by the first author, Kane, and Kohnen [3, Section 2], see Proposition 4.1 for more details as well. By [24, Theorem 1.1], the functions $\Lambda_{k+1,D}$ can be written in terms of traces of cycle integrals. Alternatively, the $\Lambda_{k+1,D}$ might be viewed “odd” positive weight analogues of the $f_{k,D}$. Recently, the $f_{k,D}$ motivated the introduction of new modular objects $\mathcal{F}_{1-k,D}$ by the first author, Kane, and Kohnen [3]. The function $\mathcal{F}_{1-k,D}$ maps to $f_{k,D}$ under the Bol operator as well as the *shadow operator* $\xi_{2-2k} := 2iv^{2-2k} \frac{\partial}{\partial \bar{\tau}}$ of Bruinier and Funke [5] (up to constants). Such a behaviour is impossible in the situation of a (globally defined) non-trivial harmonic Maass form⁶. Hence, it is natural to construct “even” analogues

⁵We explain this terminology in Section 2.

⁶One may overcome this by weakening the growth condition in Definition 2.6, see [2, Theorem 6.15].

$\Psi_{-k,D}$ of the $\mathcal{F}_{1-k,D}$ along the lines of [3]. For this, we let $\beta(x; s, w) := \int_0^x t^{s-1}(1-t)^{w-1}dt$, $x \in (0, 1]$, $\operatorname{Re}(s), \operatorname{Re}(w) > 0$, be the *incomplete β -function*, $\tau = u + iv$ throughout, and

$$\Psi_{-k,D}(\tau) := \frac{1}{2} \sum_{Q \in \mathcal{Q}_D} Q(\tau, 1)^k \beta \left(\frac{Dv^2}{|Q(\tau, 1)|^2}; k + \frac{1}{2}, \frac{1}{2} \right), \quad \tau \in \mathbb{H} \setminus E_D.$$

In the spirit of Knopp's initial preimage of $f_{k,D}$ under the Bol operator (without an additional sign change of k), it turns out that $\Psi_{-k,D}$ is a preimage of $\Lambda_{k+1,D}$ under the Bol operator and the shadow operator. If f is a cusp form of weight $2k+2$, then preimages under \mathbb{D}^{2k+1} and ξ_{-2k} , respectively, are provided by the holomorphic and non-holomorphic *Eichler integrals* (see (5.3))

$$\mathcal{E}_f(\tau) := -\frac{(2\pi i)^{2k+1}}{(2k)!} \int_{\tau}^{i\infty} f(w)(\tau-w)^{2k} dw, \quad f^*(\tau) := (2i)^{-2k-1} \int_{-\bar{\tau}}^{i\infty} \overline{f(-\bar{w})}(w+\tau)^{2k} dw. \quad (1.8)$$

To be able to insert the local cusp forms $\Lambda_{k+1,D}$ into each integral in (1.8), we work in a fundamental domain of $\operatorname{SL}_2(\mathbb{Z})$, in which we have just finitely many equivalence classes of geodesics S_Q . Integrating piecewise, both Eichler integrals of $\Lambda_{k+1,D}$ are well-defined on $\mathbb{H} \setminus E_D$. In addition we ensure in Proposition 4.4 that both Eichler integrals of $\Lambda_{k+1,D}$ exist on E_D . This established, we prove the following properties of $\Psi_{-k,D}$. We refer the reader to Subsection 2.3 for definitions.

Theorem 1.3.

- (1) *The functions $\Psi_{-k,D}$ are locally harmonic Maass forms of weight $-2k$ with continuously (however not differentially) removable singularities on E_D .*
- (2) *If $\tau \in \mathbb{H} \setminus E_D$, then we have*

$$\Psi_{-k,D}(\tau) = c_{\infty} - \frac{D^{k+\frac{1}{2}}(2k)!}{(4\pi)^{2k+1}} \mathcal{E}_{\Lambda_{k+1,D}}(\tau) + D^{k+\frac{1}{2}} \Lambda_{k+1,D}^*(\tau),$$

where

$$c_{\infty} := \frac{\pi D^{k+\frac{1}{2}}}{2^{2k}(2k+1)} \sum_{a \geq 1} \sum_{\substack{0 \leq b < 2a \\ b^2 \equiv D \pmod{4a}}} \frac{1}{a^{k+1}}.$$

Remarks.

- (1) *Using a different normalization, the constant c_{∞} was introduced in [3, (4.2), (7.3)], and can be evaluated using a result of Zagier [30, Proposition 3].*
- (2) *We prove Theorem 1.3 (2) by induction on k in Section 5, while the first author, Kane and Kohlen utilize hyperbolic expansions⁷ to prove their corresponding result [3, Theorem 1.3].*
- (3) *In joint work with Rolin [4], the authors showed that both $\mathcal{F}_{1-k,D}$ (resp. $\Psi_{-k,D}$) map to $-\mathcal{F}_{1-k,D}$ (resp. $-\Psi_{-k,D}$) under the so-called flipping operator.*

The paper is organized as follows. We recall results required for this paper in Section 2. Section 3 is devoted to the proof of Knopp's initial claims on $\psi_{k+1,D}$, to some properties of the functions $\rho_{k+1,D}$, $\varphi_{k+1,D}$, $\lambda_{k+1,D}$, and to the proofs of Theorem 1.1 as well as of Corollary 1.2. In Section 4 we investigate the behaviour of $\Lambda_{k+1,D}$, $\mathcal{E}_{\Lambda_{k+1,D}}$, and $\Lambda_{k+1,D}^*$ on E_D . Section 5 discusses the properties of $\Psi_{-k,D}$, and proves Theorem 1.3.

ACKNOWLEDGEMENTS

The authors would like to thank Caner Nazaroglu for very helpful comments and discussions. Moreover we thank the referee for very helpful comments on our manuscript. Both authors were supported by the CRC/TRR 191 ‘‘Symplectic Structures in Geometry, Algebra and Dynamics’’, funded by the DFG (project number 281071066).

⁷An excellent survey on such expansions can be found in [11] for example.

2. PRELIMINARIES

2.1. Integral binary quadratic forms and Heegner geodesics. The modular group $\mathrm{SL}_2(\mathbb{Z})$ acts on \mathcal{Q}_d by $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})\right)$

$$\left(Q \circ \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(x, y) := Q(ax + by, cx + dy).$$

The action of $\mathrm{SL}_2(\mathbb{Z})$ on \mathbb{H} is compatible with the action of $\mathrm{SL}_2(\mathbb{Z})$ on \mathcal{Q}_d , in the sense that⁸

$$(Q \circ \gamma)(\tau, 1) = j(\gamma, \tau)^2 Q(\gamma\tau, 1), \quad j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau\right) := c\tau + d. \quad (2.1)$$

Since $D > 0$ is not a square, the two roots α_Q^\pm of $Q \in \mathcal{Q}_D$ are real-quadratic and connected by the Heegner geodesic S_Q . We orientate S_Q counterclockwise (resp. clockwise) if $\mathrm{sgn}(Q) > 0$ (resp. $\mathrm{sgn}(Q) < 0$). The orientation of S_Q in turn determines the sign one catches if τ jumps across S_Q . More precisely, one has $\mathrm{sgn}(Q)\mathrm{sgn}(Q_\tau) < 0$ if and only if τ lies in the bounded component of $\mathbb{H} \setminus S_Q$. The unbounded connected component of $\mathbb{H} \setminus E_D$ is the unique such component containing $i\infty$ on its boundary. We refer the reader to the beautiful article by Duke, Imamoglu, and Tóth [8, Section 4] for more on Heegner geodesics.

We next collect results, which we utilize throughout. The following lemma is straightforward.

Lemma 2.1. *For $Q \in \mathcal{Q}_d$, $d \in \mathbb{Z}$, we have*

$$dv^2 + Q_\tau^2 v^2 = |Q(\tau, 1)|^2.$$

To determine the weights of our functions, the following lemma is useful.

Lemma 2.2. *For every $Q \in \mathcal{Q}_D$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we have*

$$(Q \circ \gamma)_\tau = Q_{\gamma\tau}, \quad \frac{\mathrm{Im}(\gamma\tau)}{|Q(\gamma\tau, 1)|} = \frac{v}{|(Q \circ \gamma)(\tau, 1)|}.$$

We also require the following elementary lemma.

Lemma 2.3. *Let $U \subseteq \mathbb{C}$ be open. Assume that $f: U \rightarrow \mathbb{C}$ is real-differentiable and satisfies $\overline{f(\bar{\tau})} = f(\tau)$. Then*

$$\overline{\frac{\partial}{\partial \bar{\tau}} f(\bar{\tau})} = \frac{\partial}{\partial \tau} f(\tau).$$

The following differentiation rules are obtained by a direct calculation.

Lemma 2.4. *Let $Q \in \mathcal{Q}_D$.*

(1) *We have*

$$v^2 \frac{\partial}{\partial \tau} Q_{-\bar{\tau}} = \frac{i}{2} Q(-\bar{\tau}, 1), \quad v^2 \frac{\partial}{\partial \tau} Q_\tau = \frac{i}{2} Q(\bar{\tau}, 1), \quad v^2 \frac{\partial}{\partial \tau} \frac{Q(\tau, 1)}{v^2} = iQ_\tau.$$

(2) *We have*

$$\frac{\partial}{\partial \bar{\tau}} \frac{v^2}{Q(\bar{\tau}, 1)} = \frac{iv^2 Q_\tau}{Q(\bar{\tau}, 1)^2}, \quad 2iv^2 \frac{\partial}{\partial \bar{\tau}} Q_\tau = Q(\tau, 1), \quad iv^2 \frac{\partial}{\partial \bar{\tau}} \frac{Q(\bar{\tau}, 1)}{v^2} = Q_\tau.$$

Letting $Q'(\tau, 1) := \frac{\partial}{\partial \tau} Q(\tau, 1)$, the following lemma can be verified by direct calculation.

Lemma 2.5. *Let $Q \in \mathcal{Q}_D$ and $\tau \in \mathbb{H}$. We have*

$$Q_\tau v + ivQ'(\tau, 1) = Q(\tau, 1), \quad Q'(\tau, 1)^2 - 2Q''(\tau, 1)Q(\tau, 1) = D.$$

⁸A good reference is for example Zagier's book [31, §8].

2.2. Maass forms and modular forms. Let $\kappa \in \frac{1}{2}\mathbb{Z}$ and d odd. Define

$$N := \begin{cases} 1 & \text{if } \kappa \in \mathbb{Z}, \\ 4 & \text{if } \kappa \in \mathbb{Z} + \frac{1}{2}, \end{cases} \quad \varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4}, \\ i & \text{if } d \equiv 3 \pmod{4}, \end{cases}$$

Let $(\frac{c}{d})$ be the extended Legendre symbol and $(\frac{a}{c} \frac{b}{d}) \in \Gamma_0(N)$. The *slash operator* is defined as

$$f|_{\kappa} \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau) := \begin{cases} (c\tau + d)^{-\kappa} f(\gamma\tau) & \text{if } \kappa \in \mathbb{Z}, \\ (\frac{c}{d}) \varepsilon_d^{2\kappa} (c\tau + d)^{-\kappa} f(\gamma\tau) & \text{if } \kappa \in \frac{1}{2} + \mathbb{Z}, \end{cases}$$

The *weight κ hyperbolic Laplace operator* is given as

$$\Delta_{\kappa} := -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + i\kappa v \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right).$$

We require various classes of modular objects.

Definition 2.6. Let $f: \mathbb{H} \rightarrow \mathbb{C}$ be a real-analytic function.

- (1) We call f a (*holomorphic*) *modular form* of weight κ for $\Gamma_0(N)$, if f satisfies the following:
 - (i) We have $f|_{\kappa}\gamma = f$ for all $\gamma \in \Gamma_0(N)$.
 - (ii) The function f is holomorphic on \mathbb{H} .
 - (iii) The function f is holomorphic at the cusps of $\Gamma_0(N)$.
- (2) We call f a *cusp form* of weight κ for $\Gamma_0(N)$, if f is a modular form that vanishes at all cusps of $\Gamma_0(N)$.
- (3) We call f a *harmonic Maass form of weight κ* for $\Gamma_0(N)$, if f satisfies the following:
 - (i) For every $\gamma \in \Gamma_0(N)$ and every $\tau \in \mathbb{H}$ we have that $f|_{\kappa}\gamma = f$.
 - (ii) The function f has eigenvalue 0 under Δ_{κ} .
 - (iii) There exists a polynomial $P_f \in \mathbb{C}[q^{-1}]$ (the principal part of f) such that

$$f(\tau) - P_f(\tau) = O(e^{-\delta v})$$

as $v \rightarrow \infty$ for some $\delta > 0$, and we require analogous conditions at all other cusps of $\Gamma_0(N)$.

Forms in *Kohnen's plus space* have the additional property that their Fourier expansion is supported on indices n satisfying $(-1)^{\kappa - \frac{1}{2}} n \equiv 0, 1 \pmod{4}$ with $\kappa \in \mathbb{Z} + \frac{1}{2}$.

We remark that Δ_{κ} splits as

$$\Delta_{\kappa} = -\xi_{2-\kappa} \circ \xi_{\kappa}, \tag{2.2}$$

which in turn implies that a harmonic Maass form naturally splits into a holomorphic and a non-holomorphic part. The operator ξ_{κ} annihilates the holomorphic part, while the Bol operator $\mathbb{D}^{1-\kappa}$, $\kappa \in -\mathbb{N}_0$, annihilates the non-holomorphic part (since our growth condition rules out a non-holomorphic constant term in the Fourier expansion). Letting $\ell \in \mathbb{N}$, the Bol operator can be written in terms of the *iterated Maass raising operator*

$$(-4\pi)^{\ell-1} \mathbb{D}^{\ell-1} = R_{2-\ell}^{\ell-1} := R_{\ell-2} \circ \dots \circ R_{2-\ell+2} \circ R_{2-\ell}, \quad R_{2-\ell}^0 := \text{id}, \quad R_{\kappa} := 2i \frac{\partial}{\partial \tau} + \frac{\kappa}{v}. \tag{2.3}$$

This identity is called *Bol's identity*, a proof can for example be found in [2, Lemma 5.3].

2.3. Locally harmonic Maass forms. In [3], so-called locally harmonic Maass forms, were introduced (for negative weights). See also [10] for the case of weight 0.

Definition 2.7 ([3, Section 2]). A function $f: \mathbb{H} \rightarrow \mathbb{C}$ is called a *locally harmonic Maass form of weight κ* with exceptional set E_D , if it obeys the following four conditions:

- (1) For every $\gamma \in \text{SL}_2(\mathbb{Z})$ we have $f|_{\kappa}\gamma = f$.
- (2) For all $\tau \in \mathbb{H} \setminus E_D$, there exists a neighborhood of τ , in which f is real-analytic and in which we have $\Delta_{\kappa}(f) = 0$.
- (3) For every $\tau \in E_D$, we have that

$$f(\tau) = \frac{1}{2} \lim_{\varepsilon \rightarrow 0^+} (f(\tau + i\varepsilon) + f(\tau - i\varepsilon)).$$

(4) The function f exhibits at most polynomial growth towards $i\infty$.

Lastly, we define the various notions of singularities appearing in this paper.

Definition 2.8. Let $f: \mathbb{H} \setminus E_D \rightarrow \mathbb{C}$.

(1) We say that f has *jumping singularities* on E_D if there exists $\tau \in E_D$ such that

$$\lim_{\varepsilon \rightarrow 0^+} (f(\tau + i\varepsilon) - f(\tau - i\varepsilon)) \in \mathbb{C} \setminus \{0\}.$$

Note that this limit depends on the geodesic S_Q on which τ is located.

(2) We say that f has *continuously removable singularities* on E_D if, for all $\tau \in E_D$,

$$\lim_{\varepsilon \rightarrow 0^+} (f(\tau + i\varepsilon) - f(\tau - i\varepsilon)) = 0.$$

(3) We say that f has *differentially removable singularities* on E_D if f is differentiable on $\mathbb{H} \setminus E_D$ and f' has continuously removable singularities on E_D .

3. PROOF OF THEOREM 1.1 AND OF COROLLARY 1.2

3.1. Knopp's claims on $\psi_{k+1,D}$. We now discuss the initial claims of Knopp on $\psi_{k+1,D}$.

Proposition 3.1.

(1) The functions $\psi_{k+1,D}$ converge absolutely on \mathbb{H} and uniformly towards $i\infty$.

(2) For $n \in \mathbb{N}$, we have

$$\mathbb{D}^{2n-1} \left(\text{Log} \left(\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+} \right) Q(\tau, 1)^{n-1} \right) = -i(2\pi)^{2n-1} (n-1)!^2 D^{n-\frac{1}{2}} \frac{1}{Q(\tau, 1)^n}.$$

(3) The functions $\psi_{k+1,D}$ satisfy $\psi_{k+1,D}(\tau + 1) = \psi_{k+1,D}(\tau)$ and (1.2).

Proof. (1) Let $Q = [a, b, c]$ and suppose that $v > 1$. Since $\alpha_Q^\pm \in \mathbb{R}$ are the zeros of Q , we have $Q(\tau, 1) = a(\tau - \alpha_Q^+)(\tau - \alpha_Q^-)$ and $v > 1$ implies that $|\tau - \alpha_Q^\pm| > 1$. Using $|a| \geq 1$ gives

$$\left| \text{Log} \left(\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+} \right) \right| \leq \left| \log \left(\frac{|Q(\tau, 1)|}{|a| |\tau - \alpha_Q^+|^2} \right) \right| + \pi \leq |\log |Q(\tau, 1)|| + \pi,$$

and (1) thus follows by the properties of $f_{\kappa,D}$ for $\kappa > 1$ (see [29]).

(2) We proceed by induction on n . The claims for $n = 1$ and $n = 2$ follow by computing

$$\frac{\partial}{\partial \tau} \text{Log} \left(\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+} \right) = -\frac{\sqrt{D}}{Q(\tau, 1)}, \quad \frac{\partial^3}{\partial \tau^3} \left(\text{Log} \left(\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+} \right) Q(\tau, 1) \right) = \frac{D^{\frac{3}{2}}}{Q(\tau, 1)^2}, \quad (3.1)$$

utilizing Lemma 2.5 for $n = 2$. To proceed with the induction step, we define for $n \in \mathbb{N}$

$$\mathfrak{f}_n(\tau) := \text{Log} \left(\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+} \right) Q(\tau, 1)^{n-1}, \quad c_n := (-1)^n (n-1)!^2.$$

Since Q is a polynomial of degree 2, we have, using the Leibniz rule,

$$\begin{aligned} \frac{\partial^{2n+1}}{\partial \tau^{2n+1}} \mathfrak{f}_{n+1}(\tau) &= \frac{\partial^{2n+1}}{\partial \tau^{2n+1}} (\mathfrak{f}_n(\tau) Q(\tau, 1)) \\ &= \mathfrak{f}_n^{(2n+1)}(\tau) Q(\tau, 1) + (2n+1) \mathfrak{f}_n^{(2n)}(\tau) Q'(\tau, 1) + (2n+1) n \mathfrak{f}_n^{(2n-1)}(\tau) Q''(\tau, 1). \end{aligned}$$

To apply the induction hypothesis, we write $\mathfrak{f}_n^{(2n)}(\tau) = \frac{\partial}{\partial \tau} \mathfrak{f}_n^{(2n-1)}(\tau)$. Combining with the second identity of Lemma 2.5 then yields

$$\frac{\partial^{2n+1}}{\partial \tau^{2n+1}} \mathfrak{f}_{n+1}(\tau) = -\frac{n^2 c_n D^{n+\frac{1}{2}}}{Q(\tau, 1)^{n+1}}.$$

Simplifying gives the claim.

(3) Translation invariance of $\psi_{k+1,D}$ follows immediately from (2.1) and the fact that

$$[a, b, c] \circ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = [a, -2a + b, a - b + c].$$

Again using (2.1) and the fact that

$$[a, b, c] \circ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = [c, -b, a],$$

we obtain that

$$\tau^{-2k-2} \psi_{k+1,D} \left(-\frac{1}{\tau} \right) - \psi_{k+1,D}(\tau) = \sum_{Q \in \mathcal{Q}_D} \frac{\text{Log} \left(\frac{-\frac{1}{\tau} - \frac{b-\sqrt{D}}{2c}}{-\frac{1}{\tau} - \frac{b+\sqrt{D}}{2c}} \right) - \text{Log} \left(\frac{\tau - \frac{b-\sqrt{D}}{2a}}{\tau - \frac{b+\sqrt{D}}{2a}} \right)}{Q(\tau, 1)^{k+1}}.$$

Next, we recall that for $z, w \in \mathbb{C} \setminus \mathbb{R}$

$$\text{Log}(z) - \text{Log}(w) = \text{Log} \left(\frac{z}{w} \right) + i \left(\text{Arg}(z) - \text{Arg}(w) - \text{Arg} \left(\frac{z}{w} \right) \right). \quad (3.2)$$

Choosing $z = \frac{-\frac{1}{\tau} - \frac{b-\sqrt{D}}{2c}}{-\frac{1}{\tau} - \frac{b+\sqrt{D}}{2c}}$, $w = \frac{\tau - \frac{b-\sqrt{D}}{2a}}{\tau - \frac{b+\sqrt{D}}{2a}}$ yields

$$\frac{z}{w} = \frac{\left(-\frac{1}{\tau} - \frac{b-\sqrt{D}}{2c} \right) \left(\tau - \frac{b+\sqrt{D}}{2a} \right)}{\left(-\frac{1}{\tau} - \frac{b+\sqrt{D}}{2c} \right) \left(\tau - \frac{b-\sqrt{D}}{2a} \right)} = \frac{\alpha_Q^+}{\alpha_Q^-} = \text{sgn}(ac) \left| \frac{\alpha_Q^+}{\alpha_Q^-} \right|. \quad (3.3)$$

Hence $\text{Arg}(z) = \text{Arg}(\text{sgn}(ac)w)$ and thus $\text{Arg}(z) - \text{Arg}(w) - \text{Arg}(\frac{z}{w})$ vanishes if $\text{sgn}(ac) = 1$. Thus the corresponding terms do not contribute to $\text{Arg}(z) - \text{Arg}(w) - \text{Arg}(\frac{z}{w})$. If $\text{sgn}(ac) = -1$, then we extend Log by its principal value $\text{Log}(x) = \log|x| + \pi i$ for $x \in \mathbb{R}^-$. Then we use that

$$\text{Arg}(-w) - \text{Arg}(w) = -\text{sgn}(\text{Im}(w))\pi, \quad (3.4)$$

and $\text{Arg}(\frac{z}{w}) = \pi$. Hence, $\text{Arg}(z) - \text{Arg}(w) - \text{Arg}(\frac{z}{w})$ vanishes if $\text{sgn}(ac) = -1$ and $\text{Im}(w) < 0$. To determine the sign of $\text{Im}(w)$, we calculate that

$$\begin{aligned} \frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+} &= \frac{\alpha_Q^+ \alpha_Q^- - (\alpha_Q^+ + \alpha_Q^-)u + u^2 + v^2}{|\tau - \alpha_Q^+|^2} - \frac{i(\alpha_Q^+ - \alpha_Q^-)v}{|\tau - \alpha_Q^+|^2} \\ &= \frac{1}{|\tau - \alpha_Q^+|^2} \left(\frac{vQ_\tau}{a} - i \frac{\sqrt{D}}{a} v \right). \end{aligned} \quad (3.5)$$

Thus, we have $\text{Im}(w) > 0$ if and only if $a < 0$. We conclude by (3.2) and (3.4) that

$$\text{Arg}(z) - \text{Arg}(w) - \text{Arg} \left(\frac{z}{w} \right) = \begin{cases} -2\pi & \text{if } a < 0 < c, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\tau^{-2k-2} \psi_{k+1,D} \left(-\frac{1}{\tau} \right) - \psi_{k+1,D}(\tau) = \sum_{Q \in \mathcal{Q}_D} \frac{\text{Log} \left(\frac{\alpha_Q^+}{\alpha_Q^-} \right)}{Q(\tau, 1)^{k+1}} - 2\pi i \sum_{\substack{Q \in \mathcal{Q}_D \\ a < 0 < c}} \frac{1}{Q(\tau, 1)^{k+1}}.$$

By mapping $Q \mapsto -Q$, we arrive at

$$\sum_{Q \in \mathcal{Q}_D} \frac{\text{Log} \left(\frac{\alpha_Q^+}{\alpha_Q^-} \right)}{Q(\tau, 1)^{k+1}} = \sum_{Q \in \mathcal{Q}_D} \frac{\log \left| \frac{\alpha_Q^+}{\alpha_Q^-} \right|}{Q(\tau, 1)^{k+1}} + \pi i \sum_{\substack{Q=[a,b,c] \in \mathcal{Q}_D \\ \text{sgn}(ac)=-1}} \frac{1}{Q(\tau, 1)^{k+1}} = \sum_{Q \in \mathcal{Q}_D} \frac{\log \left| \frac{\alpha_Q^+}{\alpha_Q^-} \right|}{Q(\tau, 1)^{k+1}}. \quad (3.6)$$

This gives the claim. \square

Remark. By (3.5) the branch cut of $\text{Log}(\frac{w-\alpha_Q^-}{w-\alpha_Q^+})$ is the interval $[\alpha_Q^-, \alpha_Q^+]$ or $[\alpha_Q^+, \alpha_Q^-]$.

3.2. The functions $\rho_{k+1,D}$, $\varphi_{k+1,D}$, and $\lambda_{k+1,D}$. Adapting the proof of Proposition 3.1 (1), (3) we deduce the following results.

Proposition 3.2. (1) The functions $\rho_{k+1,D}$ converge absolutely on $\mathbb{H} \times \mathbb{H}^-$ and uniformly as $\tau \rightarrow i\infty$ resp. $w \rightarrow -i\infty$.
(2) We have

$$\lim_{w \rightarrow -i\infty} \rho_{k+1,D}(\tau, w) = 0, \quad \lim_{\tau \rightarrow i\infty} \rho_{k+1,D}(\tau, w) = 0.$$

(3) Let $\tau \in \mathbb{H}$ and $w \in \mathbb{H}^-$. The functions $\rho_{k+1,D}$ satisfy

$$\rho_{k+1,D}(\tau+1, w+1) = \rho_{k+1,D}(\tau, w),$$

$$\tau^{-2k-2} \rho_{k+1,D}\left(-\frac{1}{\tau}, -\frac{1}{w}\right) - \rho_{k+1,D}(\tau, w) = \sum_{Q \in \mathcal{Q}_D} \frac{\log \left| \frac{\alpha_Q^+}{\alpha_Q^-} \right|}{Q(\tau, 1)^{k+1}} + 2\pi i \sum_{\substack{Q=[a,b,c] \in \mathcal{Q}_D \\ a < 0 < c}} \frac{1}{Q(\tau, 1)^{k+1}}.$$

Proof. Part (1) follows along the same lines as Proposition 3.1 (1), and part (2) is an immediate consequence of uniform continuity of $\rho_{k+1,D}$ towards $i\infty$ (together with cuspidality of the $f_{\kappa,D}$ [29]). The first assertion of part (3) follows verbatim to translation invariance of $\psi_{k+1,D}$ (see Proposition 3.1 (3)), and it remains to prove the second assertion of part (3). The idea is again an application of the rule from (3.2)

$$\text{Log}(z_1) - \text{Log}(z_2) = \text{Log}\left(\frac{z_1}{z_2}\right) + i \left(\text{Arg}(z_1) - \text{Arg}(z_2) - \text{Arg}\left(\frac{z_1}{z_2}\right) \right), \quad (3.7)$$

where $z_1 := \frac{-\frac{1}{w} - \frac{b-\sqrt{D}}{2c}}{-\frac{1}{w} - \frac{b+\sqrt{D}}{2c}}$, and $z_2 := \frac{w - \frac{-b-\sqrt{D}}{2a}}{w - \frac{-b+\sqrt{D}}{2a}}$. By (3.3), with τ replaced by w , we have

$$\frac{z_1}{z_2} = \frac{\left(-\frac{1}{w} - \frac{b-\sqrt{D}}{2c}\right) \left(w - \frac{-b+\sqrt{D}}{2a}\right)}{\left(-\frac{1}{w} - \frac{b+\sqrt{D}}{2c}\right) \left(w - \frac{-b-\sqrt{D}}{2a}\right)} = \frac{\alpha_Q^+}{\alpha_Q^-} = \text{sgn}(ac) \left| \frac{\alpha_Q^+}{\alpha_Q^-} \right|.$$

It follows that $\text{Arg}(z_1) = \text{Arg}(\text{sgn}(ac)z_2)$, and therefore $\text{Arg}(z_1) - \text{Arg}(z_2) - \text{Arg}(\frac{z_1}{z_2}) = 0$ if $\text{sgn}(ac) = 1$. If $\text{sgn}(ac) = -1$, then (3.4) yields that $\text{Arg}(-z_2) - \text{Arg}(z_2) - \text{Arg}(\frac{-z_1}{z_1}) = 0$ if $\text{Im}(z_2) < 0$ in addition. The sign of $\text{Im}(z_2)$ is given by (3.5), namely

$$z_2 = \frac{w - \frac{-b-\sqrt{D}}{2a}}{w - \frac{-b+\sqrt{D}}{2a}} = \frac{|w|^2 + \frac{b}{a} \text{Re}(w) + \frac{c}{a} - i \frac{\sqrt{D}}{a} \text{Im}(w)}{\left|w + \frac{b+\sqrt{D}}{2a}\right|^2} =: \frac{z_3}{\left|w + \frac{b+\sqrt{D}}{2a}\right|^2}.$$

Hence, we have $\text{Arg}(z_1) - \text{Arg}(z_2) - \text{Arg}(\frac{z_1}{z_2}) \neq 0$ if and only if $\text{sgn}(ac) = -1$ and $\text{sgn}(\text{Im}(z_3)) > 0$. Since $w \in \mathbb{H}^-$, we have

$$\text{sgn}(\text{Im}(z_3)) = -\text{sgn}(a) \text{sgn}(\text{Im}(w)) = \text{sgn}(a),$$

and thus we infer that

$$\left(\text{Arg}(z_1) - \text{Arg}(z_2) - \text{Arg}\left(\frac{z_1}{z_2}\right) \right) = \begin{cases} -2\pi & \text{if } c < 0 < a, \\ 0 & \text{otherwise.} \end{cases}$$

By changing $Q \mapsto -Q$ in both resulting expressions and using (3.6), we arrive at the claim. \square

We next cite Parson's [25] result on her modular integrals $\varphi_{k+1,D}$.

Lemma 3.3 ([25, Theorem 3.1]). *The functions $\varphi_{k+1,D}$ satisfy*

$$\begin{aligned} \varphi_{k+1,D}(\tau+1) &= \varphi_{k+1,D}(\tau), \\ \tau^{-2k-2} \varphi_{k+1,D}\left(-\frac{1}{\tau}\right) - \varphi_{k+1,D}(\tau) &= - \sum_{\substack{Q=[a,b,c] \in \mathcal{Q}_D \\ ac < 0}} \frac{\operatorname{sgn}(Q)}{Q(\tau,1)^{k+1}} = 2 \sum_{\substack{Q=[a,b,c] \in \mathcal{Q}_D \\ a < 0 < c}} \frac{1}{Q(\tau,1)^{k+1}}. \end{aligned}$$

Furthermore, we have

$$\lim_{\tau \rightarrow i\infty} \varphi_{k+1,D}(\tau) = 0.$$

We continue with some properties of $\lambda_{k+1,D}$.

Proposition 3.4.

- (1) *The functions $\lambda_{k+1,D}$ converge absolutely on $\mathbb{H} \times \mathbb{H}^-$, and uniformly as $\tau \rightarrow i\infty$ resp. $w \rightarrow -i\infty$.*
(2) *Let $\tau \in \mathbb{H}$ and $w \in \mathbb{H}^-$. The functions $\lambda_{k+1,D}$ are bimodular of weight $(2k+2, 0)$, that is*

$$\lambda_{k+1,D}(\tau+1, w+1) = \lambda_{k+1,D}(\tau, w), \quad \lambda_{k+1,D}\left(-\frac{1}{\tau}, -\frac{1}{w}\right) = \tau^{2k+2} \lambda_{k+1,D}(\tau, w).$$

- (3) *We have*

$$\lim_{w \rightarrow -i\infty} \lambda_{k+1,D}(\tau, w) = -2\pi i \varphi_{k+1,D}(\tau), \quad \lim_{\tau \rightarrow i\infty} \lambda_{k+1,D}(\tau, w) = 0.$$

- (4) *We have*

$$\lambda_{k+1,D}(\tau, w) = \sum_{Q \in \mathcal{Q}_D} \frac{\operatorname{Log}\left(\frac{1+i\frac{Q_w}{\sqrt{D}}}{1-i\frac{Q_w}{\sqrt{D}}}\right)}{Q(\tau,1)^{k+1}} = \sum_{Q \in \mathcal{Q}_D} \frac{\operatorname{Log}\left(-\frac{\frac{Q_w}{\sqrt{D}}-i}{\frac{Q_w}{\sqrt{D}}+i}\right)}{Q(\tau,1)^{k+1}}.$$

Proof. (1) By the definition of $\lambda_{k+1,D}$ in (1.5), we have

$$|\lambda_{k+1,D}(\tau)| \leq 2 \sum_{Q \in \mathcal{Q}_D} \frac{\left|\arctan\left(\frac{Q_w}{\sqrt{D}}\right)\right|}{|Q(\tau,1)|^{k+1}} \leq \pi \sum_{Q \in \mathcal{Q}_D} \frac{1}{|Q(\tau,1)|^{k+1}}.$$

The claim follows by the absolute convergence of the $f_{\kappa,D}$ on \mathbb{H} .

(2) Bimodularity is a direct consequence of Lemma 2.2 and (2.1).

(3) The assumption that D is not a square guarantees that the sum defining $\lambda_{k+1,D}$ runs over quadratic forms $Q = [a, b, c]$ with $ac \neq 0$. To prove the first assertion, we observe that

$$\frac{Q_w}{\sqrt{D}} \asymp a \operatorname{Im}(w) \tag{3.8}$$

as $\operatorname{Im}(w) \rightarrow -\infty$, and hence

$$\lim_{w \rightarrow -i\infty} \arctan\left(\frac{Q_w}{\sqrt{D}}\right) = -\frac{\pi}{2} \operatorname{sgn}(Q).$$

The first claim follows by the definition of $\varphi_{k+1,D}$ in (1.4). As $a \neq 0$, we have $\frac{1}{|Q(\tau,1)|^{k+1}} \rightarrow 0$ for $\tau \rightarrow i\infty$. The second claim follows by (1).

(4) The claim follows by rewriting the arctangent in (1.5) in terms of logarithms. \square

3.3. Proof of Theorem 1.1 and of Corollary 1.2. We conclude this section with the proofs of Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1. (1) This follows by combining Propositions 3.1 (3), 3.2 (3), and 3.4 (2) with Lemma 3.3.

(2) This follows by combining (1.6) with Propositions 3.2 (2) and 3.4 (2).

(3) Proposition 3.1 (1) along with

$$\lim_{\tau \rightarrow i\infty} \operatorname{Log} \left(\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+} \right) = 0 \quad (3.9)$$

implies that $\psi_{k+1,D}$ is cuspidal. By Propositions 3.2 (2), 3.4 (3), and Lemma 3.3, every function defining $\Omega_{k+1,D}$ in (1.6) is cuspidal (with respect to τ).

(4) As each function defining $\Omega_{k+1,D}$ in (1.6) is holomorphic as a function of τ , we obtain the assertion with respect to τ directly. To verify that $\Omega_{k+1,D}$ is anti-holomorphic as a function of w , we compute by Lemmas 2.1 and 2.4 (1) that

$$2i \frac{\partial}{\partial w} \arctan \left(\frac{Q_w}{\sqrt{D}} \right) = -\frac{\sqrt{D}}{Q(w, 1)}.$$

By (3.1), we deduce that

$$2i \frac{\partial}{\partial w} \arctan \left(\frac{Q_w}{\sqrt{D}} \right) = \frac{\partial}{\partial w} \operatorname{Log} \left(\frac{w - \alpha_Q^-}{w - \alpha_Q^+} \right).$$

By (1.5) and (1.6), we conclude that

$$\frac{\partial}{\partial w} \Omega_{k+1,D}(\tau, w) = 0.$$

(5) We first inspect the functions $\psi_{k+1,D} - \rho_{k+1,D}$. By (1.1) and (1.5) we have

$$\psi_{k+1,D}(\tau) - \rho_{k+1,D}(\tau, \bar{\tau}) = \sum_{Q \in \mathcal{Q}_D} \frac{\operatorname{Log} \left(\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+} \right) - \operatorname{Log} \left(\frac{\bar{\tau} - \alpha_Q^-}{\bar{\tau} - \alpha_Q^+} \right)}{Q(\tau, 1)^{k+1}}.$$

We note that

$$\operatorname{Log} \left(\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+} \right) - \operatorname{Log} \left(\frac{\bar{\tau} - \alpha_Q^-}{\bar{\tau} - \alpha_Q^+} \right) \equiv \operatorname{Log} \left(\frac{(\tau - \alpha_Q^-)(\bar{\tau} - \alpha_Q^+)}{(\tau - \alpha_Q^+)(\bar{\tau} - \alpha_Q^-)} \right) \pmod{2\pi i},$$

and we determine the multiple of $2\pi i$ now. From (3.5), we deduce that

$$\frac{(\tau - \alpha_Q^-)(\bar{\tau} - \alpha_Q^+)}{(\tau - \alpha_Q^+)(\bar{\tau} - \alpha_Q^-)} = \frac{\frac{Q_\tau}{\sqrt{D}} - i}{\frac{Q_\tau}{\sqrt{D}} + i}. \quad (3.10)$$

We use (3.2) and hence need to compute

$$\begin{aligned} & \operatorname{Log} \left(\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+} \right) - \operatorname{Log} \left(\frac{\bar{\tau} - \alpha_Q^-}{\bar{\tau} - \alpha_Q^+} \right) - \operatorname{Log} \left(\frac{(\tau - \alpha_Q^-)(\bar{\tau} - \alpha_Q^+)}{(\tau - \alpha_Q^+)(\bar{\tau} - \alpha_Q^-)} \right) \\ &= i \left(\operatorname{Arg} \left(\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+} \right) - \operatorname{Arg} \left(\frac{\bar{\tau} - \alpha_Q^-}{\bar{\tau} - \alpha_Q^+} \right) - \operatorname{Arg} \left(\frac{(\tau - \alpha_Q^-)(\bar{\tau} - \alpha_Q^+)}{(\tau - \alpha_Q^+)(\bar{\tau} - \alpha_Q^-)} \right) \right). \end{aligned} \quad (3.11)$$

Note that for $z \in \mathbb{C} \setminus \mathbb{R}$

$$\operatorname{Arg}(z) - \operatorname{Arg}(\bar{z}) - \operatorname{Arg} \left(\frac{z}{\bar{z}} \right) = \pi (1 - \operatorname{sgn}(\operatorname{Re}(z))) \operatorname{sgn}(\operatorname{Im}(z)).$$

We use this for $z = \frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+}$. By (3.5), (3.11) thus becomes $\pi i(\operatorname{sgn}(Q_\tau) - \operatorname{sgn}(Q))$. By (3.10),

$$\psi_{k+1,D}(\tau) - \rho_{k+1,D}(\tau, \bar{\tau}) = \sum_{Q \in \mathcal{Q}_D} \frac{\operatorname{Log} \left(\frac{\frac{Q_\tau}{\sqrt{D}} - i}{\frac{Q_\tau}{\sqrt{D}} + i} \right)}{Q(\tau, 1)^{k+1}} + \pi i \sum_{Q \in \mathcal{Q}_D} \frac{\operatorname{sgn}(Q_\tau) - \operatorname{sgn}(Q)}{Q(\tau, 1)^{k+1}}.$$

Combining with (1.4) gives

$$\psi_{k+1,D}(\tau) - \rho_{k+1,D}(\tau, \bar{\tau}) + 2\pi i \varphi_{k+1,D}(\tau) = \sum_{Q \in \mathcal{Q}_D} \frac{\operatorname{Log} \left(\frac{\frac{Q_\tau}{\sqrt{D}} - i}{\frac{Q_\tau}{\sqrt{D}} + i} \right)}{Q(\tau, 1)^{k+1}} + \pi i \sum_{Q \in \mathcal{Q}_D} \frac{\operatorname{sgn}(Q_\tau)}{Q(\tau, 1)^{k+1}}, \quad (3.12)$$

which is modular of weight $2k + 2$ by (2.1) and Lemma 2.2. To finish the proof, we inspect $\lambda_{k+1,D}(\tau, \bar{\tau})$. Combining $Q_{\bar{\tau}} = -Q_\tau$ with Proposition 3.4 (4) yields

$$\lambda_{k+1,D}(\tau, \bar{\tau}) = - \sum_{Q \in \mathcal{Q}_D} \frac{\operatorname{Log} \left(-\frac{\frac{Q_\tau}{\sqrt{D}} - i}{\frac{Q_\tau}{\sqrt{D}} + i} \right)}{Q(\tau, 1)^{k+1}}.$$

By (3.4), we obtain

$$\operatorname{Log} \left(\frac{\frac{Q_\tau}{\sqrt{D}} - i}{\frac{Q_\tau}{\sqrt{D}} + i} \right) - \operatorname{Log} \left(-\frac{\frac{Q_\tau}{\sqrt{D}} - i}{\frac{Q_\tau}{\sqrt{D}} + i} \right) = -\pi i \operatorname{sgn}(Q_\tau),$$

from which we conclude that

$$\operatorname{Log} \left(\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+} \right) - \operatorname{Log} \left(\frac{\bar{\tau} - \alpha_Q^-}{\bar{\tau} - \alpha_Q^+} \right) + \pi i \operatorname{sgn}(Q) + 2i \arctan \left(\frac{Q_\tau}{\sqrt{D}} \right) = 0, \quad (3.13)$$

as well as the claim using (1.6). \square

The proof of Corollary 1.2 is along the same lines as the proof of Theorem 1.1.

Proof of Corollary 1.2. Let $z \in \mathbb{H}$. Rearranging (3.13) and substituting $\tau \mapsto z$ yields

$$-\operatorname{Log} \left(\frac{\bar{z} - \alpha_Q^-}{\bar{z} - \alpha_Q^+} \right) + \pi i \operatorname{sgn}(Q) + 2i \arctan \left(\frac{Q_{\bar{z}}}{\sqrt{D}} \right) = -\operatorname{Log} \left(\frac{z - \alpha_Q^-}{z - \alpha_Q^+} \right),$$

which proves the first assertion.

Combining Propositions 3.1 (3) and 3.2 (3) shows that

$$\begin{aligned} & \tau^{-2k-2} \rho_{k+1,D} \left(-\frac{1}{\tau}, -\frac{1}{\mathfrak{z}} \right) - \rho_{k+1,D}(\tau, \mathfrak{z}) \\ &= \sum_{Q \in \mathcal{Q}_D} \frac{\log \left| \frac{\alpha_Q^+}{\alpha_Q^-} \right|}{Q(\tau, 1)^{k+1}} - 2\pi i \operatorname{sgn}(\operatorname{Im}(\mathfrak{z})) \sum_{\substack{Q=[a,b,c] \in \mathcal{Q}_D \\ a < 0 < c}} \frac{1}{Q(\tau, 1)^{k+1}}, \quad \mathfrak{z} \in \mathbb{C} \setminus \mathbb{R}. \end{aligned}$$

In other words, $\psi_{k+1,D}$ and $\rho_{k+1,D}$ transform exactly the same on $\mathbb{H} \times \mathbb{H}$, from which weight $(2k + 2, 0)$ bimodularity of $\omega_{k+1,D}$ on $\mathbb{H} \times \mathbb{H}$ immediately follows. The other assertions are immediate consequences of the definitions. \square

4. THE FUNCTION $\Lambda_{k+1,D}$

4.1. Local cusp forms. Recall the definition of $\Lambda_{k+1,D}$ in (1.7).

Remark. Let $d(z, w)$ denote the hyperbolic distance between $z, w \in \mathbb{C}$ with $\operatorname{Im}(z) \operatorname{Im}(w) > 0$. Since $D > 0$, we have (with $\tau_{[a,b,c]} := -\frac{b}{2a} + \frac{i}{2|a|}\sqrt{D}$) $\frac{Q_\tau}{\sqrt{D}} = \cosh(d(\tau, \tau_Q))$. This yields an alternative representation of $\Lambda_{k+1,D}$ as well as of $\lambda_{k+1,D}$.

We next prove our claim for $\Lambda_{k+1,D}$ from the introduction.

Proposition 4.1. *The functions $\Lambda_{k+1,D}$ are local cusp forms.*

Proof. We observe that the $\Lambda_{k+1,D}$ converge absolutely on \mathbb{H} utilizing absolute convergence of the $f_{k,D}$. We directly deduce that the $\Lambda_{k+1,D}$ are holomorphic. Using Lemma 2.2 and (2.1) shows that the $\Lambda_{k+1,D}$ are modular of weight $2k+2$. If $v > \frac{\sqrt{D}}{2}$, then $\text{sgn}(Q_\tau) = \text{sgn}(Q)$ by (3.8). Thus, cuspidality of the $\Lambda_{k+1,D}$ follows by cuspidality of the $\varphi_{k+1,D}$ (see Lemma 3.3). The local behaviour and the jumping singularities are dictated by $\text{sgn}(Q_\tau)$. \square

4.2. The local behaviour of $\Lambda_{k+1,D}$. We next provide the behaviour of $\Lambda_{k+1,D}$ on E_D .

Proposition 4.2. *If $\tau \in E_D$, then we have that*

$$\lim_{\varepsilon \rightarrow 0^+} (\Lambda_{k+1,D}(\tau + i\varepsilon) - \Lambda_{k+1,D}(\tau - i\varepsilon)) = 2 \sum_{\substack{Q \in \mathcal{Q}_D \\ Q_\tau = 0}} \frac{\text{sgn}(Q)}{Q(\tau, 1)^{k+1}}.$$

Remark. *The sum on the right-hand side is finite by [3, Lemma 5.1 (1)].*

Proof of Proposition 4.2. We adapt the proof of [3, Proposition 5.2]. We write

$$\Lambda_{k+1,D}(\tau \pm i\varepsilon) = \left(\sum_{\substack{Q \in \mathcal{Q}_D \\ Q_\tau = 0}} + \sum_{\substack{Q \in \mathcal{Q}_D \\ Q_\tau \neq 0}} \right) \frac{\text{sgn}(Q_{\tau \pm i\varepsilon})}{Q(\tau \pm i\varepsilon, 1)^{k+1}}.$$

The properties of $f_{k+1,D}$ imply that $\Lambda_{k+1,D}$ converges absolutely on \mathbb{H} and uniformly towards $i\infty$, which permits us to interchange the sums with the limit, and argue termwise.

If $Q_\tau \neq 0$, then $\tau \pm i\varepsilon$ are in the same connected component of $\mathbb{H} \setminus E_D$ for $\varepsilon > 0$ sufficiently small. Combining with [3, (5.4)], we deduce that for $\varepsilon > 0$ sufficiently small

$$\text{sgn}([a, b, c]_{\tau + i\varepsilon}) = \text{sgn}([a, b, c]_{\tau - i\varepsilon}) = \delta \text{sgn}(a),$$

where

$$\delta := \text{sgn} \left(\left| \tau + i\varepsilon + \frac{b}{2a} \right| - \frac{\sqrt{D}}{2|a|} \right) = \text{sgn} \left(\left| \tau - i\varepsilon + \frac{b}{2a} \right| - \frac{\sqrt{D}}{2|a|} \right) = \pm 1.$$

Thus

$$\lim_{\varepsilon \rightarrow 0^+} \left(\frac{\text{sgn}(Q_{\tau + i\varepsilon})}{Q(\tau + i\varepsilon, 1)^{k+1}} - \frac{\text{sgn}(Q_{\tau - i\varepsilon})}{Q(\tau - i\varepsilon, 1)^{k+1}} \right) = \delta \lim_{\varepsilon \rightarrow 0^+} \left(\frac{\text{sgn}(Q)}{Q(\tau + i\varepsilon, 1)^{k+1}} - \frac{\text{sgn}(Q)}{Q(\tau - i\varepsilon, 1)^{k+1}} \right) = 0.$$

If $Q_\tau = 0$, then $\tau \pm i\varepsilon$ are in different connected components of $\mathbb{H} \setminus E_D$ for all $\varepsilon > 0$. This is justified by [3, (5.6)], namely, for every $\varepsilon > 0$,

$$\left| \tau - i\varepsilon + \frac{b}{2a} \right| - \frac{\sqrt{D}}{2|a|} < \left| \tau + \frac{b}{2a} \right| - \frac{\sqrt{D}}{2|a|} = 0 < \left| \tau + i\varepsilon + \frac{b}{2a} \right| - \frac{\sqrt{D}}{2|a|}.$$

Combining with [3, (5.4)] implies that $\text{sgn}(Q_{\tau \pm i\varepsilon}) = \pm \text{sgn}(Q)$, and consequently

$$\lim_{\varepsilon \rightarrow 0^+} \left(\frac{\text{sgn}(Q_{\tau + i\varepsilon})}{Q(\tau + i\varepsilon, 1)^{k+1}} - \frac{\text{sgn}(Q_{\tau - i\varepsilon})}{Q(\tau - i\varepsilon, 1)^{k+1}} \right) = 2 \frac{\text{sgn}(Q)}{Q(\tau, 1)^{k+1}}. \quad \square$$

We next inspect the sum appearing in Proposition 4.2.

Lemma 4.3. *The sum*

$$\sum_{\substack{Q \in \mathcal{Q}_D \\ Q_\tau = 0}} \frac{\text{sgn}(Q)}{Q(\tau, 1)^{k+1}}$$

does not vanish identically on E_D .

Proof. Let $\tau \in E_D$. Then we have $\tau \in S_\Omega$ for some $\Omega \in \mathcal{Q}_D$. On the one hand, the sum in the lemma has a pole of order $k+1 > 0$ at α_Ω^\pm , and hence both limits

$$\lim_{\substack{\tau \rightarrow \alpha_\Omega^\pm \\ \tau \in S_\Omega}} \left| \sum_{\substack{Q \in \mathcal{Q}_D \\ Q_\tau = 0}} \frac{\text{sgn}(Q)}{Q(\tau, 1)^{k+1}} \right|$$

tend towards ∞ . On the other hand, the sum is continuous on S_Ω , and the contribution from the terms corresponding to $Q \neq \Omega$ is finite at α_Ω^\pm . \square

4.3. The local behaviour of $\mathcal{E}_{\Lambda_{k+1,D}}$ and $\Lambda_{k+1,D}^*$. We next prove that the Eichler integrals of $\Lambda_{k+1,D}$ exist on E_D .

Proposition 4.4. *Let $\tau \in E_D$. Then we have*

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \left(\mathcal{E}_{\Lambda_{k+1,D}}(\tau + i\varepsilon) - \mathcal{E}_{\Lambda_{k+1,D}}(\tau - i\varepsilon) \right) &= -\frac{2(2\pi i)^{2k+1}}{(2k)!} \int_0^{i\infty} \sum_{\substack{Q \in \mathcal{Q}_D \\ Q_{\tau+w}=0}} \frac{\text{sgn}(Q)}{Q(\tau+w, 1)^{k+1}} w^{2k} dw, \\ \lim_{\varepsilon \rightarrow 0^+} \left(\Lambda_{k+1,D}^*(\tau + i\varepsilon) - \Lambda_{k+1,D}^*(\tau - i\varepsilon) \right) &= -\frac{2}{(2i)^{2k+1}} \int_{2iv}^{i\infty} \sum_{\substack{Q \in \mathcal{Q}_D \\ Q_{\tau-w}=0}} \frac{\text{sgn}(Q)}{Q(\tau-w, 1)^{k+1}} w^{2k} dw. \end{aligned}$$

Remark. If $\tau + w \notin E_D$ resp. $\tau - w \notin E_D$, then the sums inside the integrands on the right-hand sides of Proposition 4.4 are empty. If $\tau + w \in E_D$ resp. $\tau - w \in E_D$, then the sums inside the integrands are finite as remarked after Proposition 4.2. Thus, each integral runs over a bounded domain, because the integrands vanish as soon as⁹ $\text{Im}(\tau + w) > \frac{\sqrt{D}}{2}$ or $\tau - w$ moves out of \mathbb{H} . Hence, the integrals on the right-hand side of Proposition 4.4 exist.

Proof of Proposition 4.4. As $\tau \pm i\varepsilon \notin E_D$ for $\varepsilon > 0$, we utilize (1.7). Changing variables gives

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0^+} \left(\mathcal{E}_{\Lambda_{k+1,D}}(\tau + i\varepsilon) - \mathcal{E}_{\Lambda_{k+1,D}}(\tau - i\varepsilon) \right) \\ &= -\frac{(2\pi i)^{2k+1}}{(2k)!} \lim_{\varepsilon \rightarrow 0^+} \int_0^{i\infty} \sum_{Q \in \mathcal{Q}_D} \left(\frac{\text{sgn}(Q_{\tau+i\varepsilon+w})}{Q(\tau+i\varepsilon+w, 1)^{k+1}} - \frac{\text{sgn}(Q_{\tau-i\varepsilon+w})}{Q(\tau-i\varepsilon+w, 1)^{k+1}} \right) w^{2k} dw, \\ &\lim_{\varepsilon \rightarrow 0^+} \left(\Lambda_{k+1,D}^*(\tau + i\varepsilon) - \Lambda_{k+1,D}^*(\tau - i\varepsilon) \right) = \frac{1}{(2i)^{2k+1}} \\ &\times \lim_{\varepsilon \rightarrow 0^+} \left(\int_{2i(v-\varepsilon)}^{i\infty} \sum_{Q \in \mathcal{Q}_D} \frac{\text{sgn}(Q_{\tau-w-i\varepsilon}) w^{2k}}{Q(\tau-w-i\varepsilon, 1)^{k+1}} dw - \int_{2i(v+\varepsilon)}^{i\infty} \sum_{Q \in \mathcal{Q}_D} \frac{\text{sgn}(Q_{\tau-w+i\varepsilon}) w^{2k}}{Q(\tau-w+i\varepsilon, 1)^{k+1}} dw \right), \end{aligned}$$

where we use $Q_{\bar{z}} = -Q_z$ for $\Lambda_{k+1,D}^*$. We next justify interchanging the limit $\varepsilon \rightarrow 0^+$ with the holomorphic Eichler integral. By (1.7), $\Lambda_{k+1,D}$ vanishes at $i\infty$, and converges uniformly towards $i\infty$ as the sign-function is bounded (as $f_{\kappa,D}$ converges uniformly towards $i\infty$ for $\kappa > 1$). By modularity of $\Lambda_{k+1,D}$, both assertions hold towards 0 as well. So the integral converges uniformly, and this permits the exchange of the limit $\varepsilon \rightarrow 0^+$ with the integral.

We consider the holomorphic Eichler integral first. If $\tau + w \notin E_D$, then the limit inside the integral vanishes, because $\tau + w + i\varepsilon$ and $\tau + w - i\varepsilon$ are in the same connected component for ε sufficiently small. If $\tau + w \in E_D$, then we apply Proposition 4.2 to obtain

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0^+} \left(\mathcal{E}_{\Lambda_{k+1,D}}(\tau + i\varepsilon) - \mathcal{E}_{\Lambda_{k+1,D}}(\tau - i\varepsilon) \right) \\ &= -\frac{(2\pi i)^{2k+1}}{(2k)!} \int_0^{i\infty} \lim_{\varepsilon \rightarrow 0^+} (\Lambda_{k+1,D}(\tau + w + i\varepsilon) - \Lambda_{k+1,D}(\tau + w - i\varepsilon)) w^{2k} dw \\ &= -\frac{2(2\pi i)^{2k+1}}{(2k)!} \int_0^{i\infty} \sum_{\substack{Q \in \mathcal{Q}_D \\ Q_{\tau+w}=0}} \frac{\text{sgn}(Q)}{Q(\tau+w, 1)^{k+1}} w^{2k} dw. \end{aligned}$$

⁹If $\text{Im}(\tau + w) > \frac{\sqrt{D}}{2}$, then $\tau + w$ lies in the unbounded component of $\mathbb{H} \setminus E_D$.

Now, we treat the non-holomorphic Eichler integrals, and first split one of them as

$$\begin{aligned} \int_{2i(v-\varepsilon)}^{i\infty} \sum_{Q \in \mathcal{Q}_D} \frac{\operatorname{sgn}(Q_{\tau-w-i\varepsilon})}{Q(\tau-w-i\varepsilon, 1)^{k+1}} w^{2k} dw \\ = \left(\int_{2i(v-\varepsilon)}^{2i(v+\varepsilon)} + \int_{2i(v+\varepsilon)}^{i\infty} \right) \sum_{Q \in \mathcal{Q}_D} \frac{\operatorname{sgn}(Q_{\tau-w-i\varepsilon})}{Q(\tau-w-i\varepsilon, 1)^{k+1}} w^{2k} dw. \end{aligned}$$

We note that

$$\lim_{\varepsilon \rightarrow 0^+} \int_{2i(v-\varepsilon)}^{2i(v+\varepsilon)} \frac{\operatorname{sgn}(Q_{\tau-w-i\varepsilon})}{Q(\tau-w-i\varepsilon, 1)^{k+1}} w^{2k} dw = 0,$$

because the integrand is bounded in the domain of integration, which has measure 0 as $\varepsilon \rightarrow 0^+$. Hence, it remains to consider the integral from $2i(v+\varepsilon)$ to $i\infty$. If $\tau-w \notin E_D$, then we have

$$\lim_{\varepsilon \rightarrow 0^+} \int_{2i(v+\varepsilon)}^{i\infty} \left(-\frac{\operatorname{sgn}(Q_{\tau-w+i\varepsilon})}{Q(\tau-w+i\varepsilon, 1)^{k+1}} + \frac{\operatorname{sgn}(Q_{\tau-w-i\varepsilon})}{Q(\tau-w-i\varepsilon, 1)^{k+1}} \right) w^{2k} dw = 0,$$

as in the previous case, because $\tau-w \pm i\varepsilon$ are in the same connected component for ε sufficiently small. If $\tau-w \in E_D$, then we obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \int_{2i(v+\varepsilon)}^{i\infty} \sum_{Q \in \mathcal{Q}_D} \left(-\frac{\operatorname{sgn}(Q_{\tau-w+i\varepsilon})}{Q(\tau-w+i\varepsilon, 1)^{k+1}} + \frac{\operatorname{sgn}(Q_{\tau-w-i\varepsilon})}{Q(\tau-w-i\varepsilon, 1)^{k+1}} \right) w^{2k} dw \\ = -2 \int_{2iv}^{i\infty} \sum_{\substack{Q \in \mathcal{Q}_D \\ Q_{\tau-w}=0}} \frac{\operatorname{sgn}(Q)}{Q(\tau-w, 1)^{k+1}} w^{2k} dw \end{aligned}$$

by Proposition 4.2 exactly as in the previous case. \square

5. THE FUNCTION $\Psi_{-k,D}$ AND THE PROOF OF THEOREM 1.3

5.1. Convergence of $\Psi_{-k,D}$. We first establish convergence of $\Psi_{-k,D}$.

Proposition 5.1. *The sum defining $\Psi_{-k,D}$ converges compactly on $\mathbb{H} \setminus E_D$, and does not converge on E_D .*

Proof. If $\tau \in \mathbb{H} \setminus E_D$, then $\operatorname{sgn}(Q_\tau) = \pm 1$ and thus the claim follows directly by [3, Proposition 4.1] after summing over all narrow equivalence classes there. (The class number of positive discriminants is finite.) If $\tau \in E_D$, then the incomplete β -function reduces to a constant depending only on k according to Lemma 2.1. Hence, the sum defining $\Psi_{-k,D}$ does not converge on E_D as the sum is infinite and $\beta(1; k + \frac{1}{2}, \frac{1}{2}) \neq 0$. \square

5.2. Behaviour of $\Psi_{-k,D}$ under differentiation. We inspect the behaviour of $\Psi_{-k,D}$ under differential operators.

Proposition 5.2. *Let $\tau \in \mathbb{H} \setminus E_D$.*

(1) *We have*

$$\xi_{-2k}(\Psi_{-k,D}(\tau)) = D^{k+\frac{1}{2}} \Lambda_{k+1,D}(\tau).$$

(2) *We have*

$$\mathbb{D}^{2k+1}(\Psi_{-k,D}(\tau)) = -\frac{D^{k+\frac{1}{2}}(2k)!}{(4\pi)^{2k+1}} \Lambda_{k+1,D}(\tau).$$

(3) *We have*

$$\Delta_{-2k}(\Psi_{-k,D}(\tau)) = 0.$$

Define

$$g_n^{[1]}(\tau) := Q(\tau, 1)^n \beta \left(\frac{Dv^2}{|Q(\tau, 1)|^2}; n + \frac{1}{2}, \frac{1}{2} \right), \quad n \in \mathbb{N}_0.$$

The proof of Proposition 5.2 is based on the following three technical lemmas.

Lemma 5.3. *We have for $n \in \mathbb{N}_0$*

$$g_{n+1}^{[1]}(\tau) = \frac{n + \frac{1}{2}}{n + 1} Q(\tau, 1) g_n^{[1]}(\tau) - \frac{D^{n+\frac{1}{2}} v^{2n+2} |Q_\tau|}{n + 1} \frac{1}{Q(\bar{\tau}, 1)^{n+1}}.$$

Proof. By [1, (8.17.20)], we have that

$$\frac{\beta(x; a, b)}{\beta(1; a, b)} = \frac{\beta(x; a + 1, b)}{\beta(1; a + 1, b)} + \frac{x^a(1 - x)^b}{a\beta(1; a, b)}.$$

This gives that

$$\begin{aligned} & \beta\left(\frac{Dv^2}{|Q(\tau, 1)|^2}; n + \frac{3}{2}, \frac{1}{2}\right) \\ &= \frac{\beta\left(1; n + \frac{3}{2}, \frac{1}{2}\right)}{\beta\left(1; n + \frac{1}{2}, \frac{1}{2}\right)} \left(\beta\left(\frac{Dv^2}{|Q(\tau, 1)|^2}; n + \frac{1}{2}, \frac{1}{2}\right) - \frac{\left(\frac{Dv^2}{|Q(\tau, 1)|^2}\right)^{n+\frac{1}{2}} \left(1 - \frac{Dv^2}{|Q(\tau, 1)|^2}\right)^{\frac{1}{2}}}{n + \frac{1}{2}} \right). \end{aligned}$$

Using Lemma 2.1, we compute

$$\left(\frac{Dv^2}{|Q(\tau, 1)|^2}\right)^{n+\frac{1}{2}} \left(1 - \frac{Dv^2}{|Q(\tau, 1)|^2}\right)^{\frac{1}{2}} = \frac{D^{n+\frac{1}{2}} v^{2n+2} |Q_\tau|}{|Q(\tau, 1)|^{2n+2}},$$

and since $\frac{\beta(1; n+\frac{3}{2}, \frac{1}{2})}{\beta(1; n+\frac{1}{2}, \frac{1}{2})} = \frac{n+\frac{1}{2}}{n+1}$, we obtain the claim. \square

Lemma 5.3 motivates to define the auxiliary function

$$g_n^{[2]}(\tau) := \frac{D^{n-\frac{1}{2}} v^{2n} |Q_\tau|}{Q(\bar{\tau}, 1)^n}.$$

The second technical lemma treats the image of $g_{n+1}^{[2]}$ under differentiation.

Lemma 5.4. *We have for $n \in \mathbb{N}$*

$$\frac{\partial^{2n+1}}{\partial \tau^{2n+1}} g_n^{[2]}(\tau) = 0.$$

Proof. We prove the claim by induction. If $n = 1$, then the claim follows by applying Lemma 2.4 (1) three times. For the induction step, Lemma 2.4 (1) yields that

$$\frac{\partial}{\partial \tau} (v^{\ell+2} Q_\tau) = -\frac{i}{2} \ell v^{\ell+1} Q_\tau + \frac{i}{2} v^\ell Q(\bar{\tau}, 1)$$

for every $\ell \in \mathbb{N}_0$. Noting that $\frac{\partial^{\ell+1}}{\partial \tau^{\ell+1}} (v^\ell Q(\bar{\tau}, 1)) = 0$, we obtain

$$\frac{\partial^{\ell+2}}{\partial \tau^{\ell+2}} (v^{\ell+1} Q_\tau) = -\frac{i}{2} (\ell + 1) \frac{\partial^{\ell+1}}{\partial \tau^{\ell+1}} (v^\ell Q_\tau).$$

Consequently, we find that

$$\frac{\partial^{2n+3}}{\partial \tau^{2n+3}} g_{n+1}^{[2]}(\tau) = -\frac{D(2n+2)(2n+1)}{4Q(\bar{\tau}, 1)} \frac{\partial^{2n+1}}{\partial \tau^{2n+1}} g_n^{[2]}(\tau).$$

The right-hand side vanishes by the induction hypothesis, as desired. \square

The third lemma contains the main technical claim.

Lemma 5.5. *We have for $n \in \mathbb{N}_0$*

$$\frac{\partial^{2n+1}}{\partial \tau^{2n+1}} g_n^{[1]}(\tau) = \frac{i(-1)^{n+1} D^{n+\frac{1}{2}} (2n)! \operatorname{sgn}(Q_\tau)}{2^{2n} Q(\tau, 1)^{n+1}}.$$

Proof. We prove the lemma by induction.

Step 1: The case $n = 0$

We apply the Fundamental Theorem of Calculus, Lemma 2.1, and Lemma 2.4, yielding

$$\frac{\partial}{\partial \tau} \beta \left(\frac{Dv^2}{|Q(\tau, 1)|^2}; n + \frac{1}{2}, \frac{1}{2} \right) = -\frac{iD^{n+\frac{1}{2}}v^{2n} \operatorname{sgn}(Q_\tau)}{|Q(\tau, 1)|^{2n} Q(\tau, 1)} \quad (5.1)$$

for every $n \in \mathbb{N}_0$. In particular, this proves the desired identity for $n = 0$.

Step 2: The case $n = 1$

Using (5.1) and the first identity of Lemma 2.5, we compute that

$$R_{-2n} \left(g_n^{[1]}(\tau) \right) = -2nQ_\tau \frac{g_n^{[1]}(\tau)}{Q(\tau, 1)} + \frac{2D^{n+\frac{1}{2}}v^{2n} \operatorname{sgn}(Q_\tau)}{Q(\bar{\tau}, 1)^n Q(\tau, 1)}.$$

Lemma 5.3 with $n \mapsto n - 1$ gives

$$\frac{g_n^{[1]}(\tau)}{Q(\tau, 1)} = \frac{n - \frac{1}{2}}{n} g_{n-1}^{[1]}(\tau) - \frac{D^{n-\frac{1}{2}}v^{2n} \operatorname{sgn}(Q_\tau) Q_\tau}{n Q(\bar{\tau}, 1)^n Q(\tau, 1)}.$$

Plugging into the previous equation and applying Lemma 2.1 yields

$$R_{-2n} \left(g_n^{[1]}(\tau) \right) = -(2n - 1)Q_\tau g_{n-1}^{[1]}(\tau) + \frac{2D^{n-\frac{1}{2}}v^{2n-2} \operatorname{sgn}(Q_\tau)}{Q(\bar{\tau}, 1)^{n-1}}.$$

We compute

$$\begin{aligned} R_{2-2n} \left(\frac{2D^{n-\frac{1}{2}}v^{2n-2} \operatorname{sgn}(Q_\tau)}{Q(\bar{\tau}, 1)^{n-1}} \right) &= 0, \\ R_{2-2n} \left(Q_\tau g_{n-1}^{[1]}(\tau) \right) &= Q_\tau R_{2-2n} \left(g_{n-1}^{[1]}(\tau) \right) - g_{n-1}^{[1]}(\tau) \frac{Q(\bar{\tau}, 1)}{v^2} \end{aligned}$$

by Lemma 2.4 (1). We infer that

$$R_{2-2n} \circ R_{-2n} \left(g_n^{[1]}(\tau) \right) = -(2n - 1) \left(Q_\tau R_{2-2n} \left(g_{n-1}^{[1]}(\tau) \right) - g_{n-1}^{[1]}(\tau) \frac{Q(\bar{\tau}, 1)}{v^2} \right).$$

Now, we suppose that $n = 1$. Then the previous equation gives

$$R_0 \circ R_{-2} \left(g_1^{[1]}(\tau) \right) = -Q_\tau R_0 \left(g_0^{[1]}(\tau) \right) + g_0^{[1]}(\tau) \frac{Q(\bar{\tau}, 1)}{v^2}.$$

We then compute, using (5.1)

$$R_0 \left(g_0^{[1]}(\tau) \right) = 2i \frac{\partial}{\partial \tau} \beta \left(\frac{Dv^2}{|Q(\tau, 1)|^2}; \frac{1}{2}, \frac{1}{2} \right) = \frac{2D^{\frac{1}{2}} \operatorname{sgn}(Q_\tau)}{Q(\tau, 1)}.$$

Combining this with the previous equation we obtain

$$R_2 \circ R_0 \circ R_{-2} \left(g_1^{[1]}(\tau) \right) = R_2 \left(-2Q_\tau \frac{D^{\frac{1}{2}} \operatorname{sgn}(Q_\tau)}{Q(\tau, 1)} + \beta \left(\frac{Dv^2}{|Q(\tau, 1)|^2}; \frac{1}{2}, \frac{1}{2} \right) \frac{Q(\bar{\tau}, 1)}{v^2} \right).$$

By Lemma 2.4 (1) and (5.1), we calculate that

$$\begin{aligned} \frac{\partial}{\partial \tau} \left(-2Q_\tau \frac{D^{\frac{1}{2}} \operatorname{sgn}(Q_\tau)}{Q(\tau, 1)} + \beta \left(\frac{Dv^2}{|Q(\tau, 1)|^2}; \frac{1}{2}, \frac{1}{2} \right) \frac{Q(\bar{\tau}, 1)}{v^2} \right) &= -\frac{iQ(\bar{\tau}, 1) D^{\frac{1}{2}} \operatorname{sgn}(Q_\tau)}{v^2 Q(\tau, 1)} \\ &+ 2Q_\tau \frac{D^{\frac{1}{2}} \operatorname{sgn}(Q_\tau)}{Q(\tau, 1)^2} Q'(\tau, 1) - \frac{iD^{\frac{1}{2}} \operatorname{sgn}(Q_\tau) Q(\bar{\tau}, 1)}{Q(\tau, 1) v^2} + \frac{i}{v^3} \beta \left(\frac{Dv^2}{|Q(\tau, 1)|^2}; \frac{1}{2}, \frac{1}{2} \right) Q(\bar{\tau}, 1). \end{aligned}$$

Using Lemma 2.1 and the first identity of Lemma 2.5, we obtain

$$\begin{aligned}
R_2 \circ R_0 \circ R_{-2} \left(g_1^{[1]}(\tau) \right) &= 2i \left(-2i \frac{Q(\bar{\tau}, 1)}{v^2} \frac{\sqrt{D} \operatorname{sgn}(Q_\tau)}{Q(\tau, 1)} + 2 \frac{\sqrt{D} |Q_\tau| Q'(\tau, 1)}{Q(\tau, 1)^2} \right. \\
&\quad \left. + i\beta \left(\frac{Dv^2}{|Q(\tau, 1)|^2}; \frac{1}{2}, \frac{1}{2} \right) \frac{Q(\bar{\tau}, 1)}{v^3} \right) + \frac{2}{v} \left(-\frac{2\sqrt{D} |Q_\tau|}{Q(\tau, 1)} + \beta \left(\frac{Dv^2}{|Q(\tau, 1)|^2}; \frac{1}{2}, \frac{1}{2} \right) \frac{Q(\bar{\tau}, 1)}{v^2} \right) \\
&= 4\sqrt{D} \frac{|Q(\tau, 1)|^2 \operatorname{sgn}(Q_\tau)}{v^2 Q(\tau, 1)^2} - \frac{4\sqrt{D} |Q_\tau|}{Q(\tau, 1)^2} \left(-iQ'(\tau, 1) + \frac{Q(\tau, 1)}{v} \right) \\
&= 4\sqrt{D} \left(D + Q_\tau^2 \right) \frac{\operatorname{sgn}(Q_\tau)}{Q(\tau, 1)^2} - \frac{4\sqrt{D} \operatorname{sgn}(Q_\tau)}{Q(\tau, 1)^2} Q_\tau^2 = \frac{4D^{\frac{3}{2}} \operatorname{sgn}(Q_\tau)}{Q(\tau, 1)^2}.
\end{aligned}$$

We can then directly conclude the claim using Bol's identity (2.3).

Step 3: Application of Lemmas 5.3 and 5.4 and reducing to $2n + 2$ derivatives

Employing Lemma 5.3 and Lemma 5.4 with $n \mapsto n + 1$ yields

$$\frac{\partial^{2n+3}}{\partial \tau^{2n+3}} g_{n+1}^{[1]}(\tau) = \frac{n + \frac{1}{2}}{n + 1} \frac{\partial^{2n+3}}{\partial \tau^{2n+3}} \left(Q(\tau, 1) g_n^{[1]}(\tau) \right). \quad (5.2)$$

By (5.1), we compute that

$$\frac{\partial}{\partial \tau} \left(Q(\tau, 1) g_n^{[1]}(\tau) \right) = (n + 1) Q(\tau, 1)^n Q'(\tau, 1) \beta \left(\frac{Dv^2}{|Q(\tau, 1)|^2}; n + \frac{1}{2}, \frac{1}{2} \right) - \frac{iD^{n+\frac{1}{2}} v^{2n} \operatorname{sgn}(Q_\tau)}{Q(\bar{\tau}, 1)^n}.$$

We observe that the final term gets annihilated by differentiating $2n + 1$ times and thus

$$\frac{n + \frac{1}{2}}{n + 1} \frac{\partial^{2n+3}}{\partial \tau^{2n+3}} \left(Q(\tau, 1) g_n^{[1]}(\tau) \right) = \left(n + \frac{1}{2} \right) \frac{\partial^{2n+2}}{\partial \tau^{2n+2}} \left(Q(\tau, 1)^n Q'(\tau, 1) \beta \left(\frac{Dv^2}{|Q(\tau, 1)|^2}; n + \frac{1}{2}, \frac{1}{2} \right) \right).$$

Step 4: Reducing to $2n + 1$ derivatives

By (5.1), we furthermore calculate that

$$\begin{aligned}
\frac{\partial}{\partial \tau} \left(Q(\tau, 1)^n Q'(\tau, 1) \beta \left(\frac{Dv^2}{|Q(\tau, 1)|^2}; n + \frac{1}{2}, \frac{1}{2} \right) \right) &= Q(\tau, 1)^n Q''(\tau, 1) \beta \left(\frac{Dv^2}{|Q(\tau, 1)|^2}; n + \frac{1}{2}, \frac{1}{2} \right) \\
&\quad + n Q(\tau, 1)^{n-1} Q'(\tau, 1)^2 \beta \left(\frac{Dv^2}{|Q(\tau, 1)|^2}; n + \frac{1}{2}, \frac{1}{2} \right) - i Q(\tau, 1)^n Q'(\tau, 1) \frac{D^{n+\frac{1}{2}} v^{2n} \operatorname{sgn}(Q_\tau)}{|Q(\tau, 1)|^{2n} Q(\tau, 1)}.
\end{aligned}$$

By the first identity of Lemma 2.5, the final term may be rewritten as

$$-i Q(\tau, 1)^n Q'(\tau, 1) \frac{D^{n+\frac{1}{2}} v^{2n} \operatorname{sgn}(Q_\tau)}{|Q(\tau, 1)|^{2n} Q(\tau, 1)} = -\frac{D^{n+\frac{1}{2}} v^{2n-1} \operatorname{sgn}(Q_\tau)}{Q(\bar{\tau}, 1)^n} + \frac{D^{n+\frac{1}{2}} v^{2n} |Q_\tau|}{Q(\bar{\tau}, 1)^n Q(\tau, 1)}.$$

Again the final term gets annihilated upon differentiating $2n + 1$ times. Consequently, we obtain, by the second identity of Lemma 2.5,

$$\begin{aligned}
\frac{n + \frac{1}{2}}{n + 1} \frac{\partial^{2n+3}}{\partial \tau^{2n+3}} \left(Q(\tau, 1) g_n^{[1]}(\tau) \right) &= \left(n + \frac{1}{2} \right) \frac{\partial^{2n+1}}{\partial \tau^{2n+1}} \left(Dn Q(\tau, 1)^{n-1} \beta \left(\frac{Dv^2}{|Q(\tau, 1)|^2}; n + \frac{1}{2}, \frac{1}{2} \right) \right. \\
&\quad \left. + (2n + 1) Q(\tau, 1)^n Q''(\tau, 1) \beta \left(\frac{Dv^2}{|Q(\tau, 1)|^2}; n + \frac{1}{2}, \frac{1}{2} \right) + \frac{D^{n+\frac{1}{2}} v^{2n} |Q_\tau|}{Q(\bar{\tau}, 1)^n Q(\tau, 1)} \right) \\
&= \left(n + \frac{1}{2} \right) \frac{\partial^{2n+1}}{\partial \tau^{2n+1}} \left(Dn \frac{g_n^{[1]}(\tau)}{Q(\tau, 1)} + (2n + 1) Q''(\tau, 1) g_n^{[1]}(\tau) + \frac{D g_n^{[2]}(\tau)}{Q(\tau, 1)} \right).
\end{aligned}$$

Step 5: Application of the induction hypothesis

We use Lemma 5.3 with $n \mapsto n - 1$, to obtain

$$\frac{g_n^{[1]}(\tau)}{Q(\tau, 1)} = \frac{n - \frac{1}{2}}{n} g_{n-1}^{[1]}(\tau) - \frac{g_n^{[2]}(\tau)}{n Q(\tau, 1)},$$

and hence, using step 4,

$$\begin{aligned} \frac{n + \frac{1}{2}}{n + 1} \frac{\partial^{2n+3}}{\partial \tau^{2n+3}} \left(Q(\tau, 1) g_n^{[1]}(\tau) \right) \\ = \left(n + \frac{1}{2} \right) \frac{\partial^{2n+1}}{\partial \tau^{2n+1}} \left(D \left(n - \frac{1}{2} \right) g_{n-1}^{[1]}(\tau) + (2n + 1) Q''(\tau, 1) g_n^{[1]}(\tau) \right). \end{aligned}$$

The induction hypothesis for n and $n - 1$, and the fact that $Q''(\tau, 1)$ is independent of τ gives

$$\begin{aligned} \frac{n + \frac{1}{2}}{n + 1} \frac{\partial^{2n+3}}{\partial \tau^{2n+3}} \left(Q(\tau, 1) g_n^{[1]}(\tau) \right) \\ = \frac{(-1)^n i D^{n+\frac{1}{2}} \left(n + \frac{1}{2} \right) (2n)! \operatorname{sgn}(Q_\tau)}{4^n} \left(\frac{1}{n} \frac{\partial^2}{\partial \tau^2} \frac{1}{Q(\tau, 1)^n} - (2n + 1) \frac{Q''(\tau, 1)}{Q(\tau, 1)^{n+1}} \right). \end{aligned}$$

Step 6: Simplifying the expressions

Using the second identity of Lemma 2.5, we compute

$$\frac{1}{n} \frac{\partial^2}{\partial \tau^2} \frac{1}{Q(\tau, 1)^n} - (2n + 1) \frac{Q''(\tau, 1)}{Q(\tau, 1)^{n+1}} = \frac{D(n + 1)}{Q(\tau, 1)^{n+2}}.$$

Inserting this into the result from step 5 yields

$$\frac{n + \frac{1}{2}}{n + 1} \frac{\partial^{2n+3}}{\partial \tau^{2n+3}} \left(Q(\tau, 1) g_n^{[1]}(\tau) \right) = \frac{(-1)^n i \left(n + \frac{1}{2} \right) (n + 1) (2n)! D^{n+\frac{3}{2}} \operatorname{sgn}(Q_\tau)}{4^n Q(\tau, 1)^{n+2}}.$$

By (5.2), we ultimately arrive at the claim of the lemma (with $n \mapsto n + 1$). \square

We are now ready to prove Proposition 5.2.

Proof of Proposition 5.2. (1) By Lemma 2.3 and (5.1), we obtain

$$\frac{\partial}{\partial \bar{\tau}} \beta \left(\frac{Dv^2}{|Q(\tau, 1)|^2}; k + \frac{1}{2}, \frac{1}{2} \right) = \frac{i D^{k+\frac{1}{2}} v^{2k} \operatorname{sgn}(Q_\tau)}{|Q(\tau, 1)|^{2k} Q(\bar{\tau}, 1)}.$$

This implies the claim.

(2) Lemma 5.5 implies that

$$\frac{1}{2} \mathbb{D}^{2k+1} \left(g_k^{[1]}(\tau) \right) = - \frac{D^{k+\frac{1}{2}} (2k)! \operatorname{sgn}(Q_\tau)}{(4\pi)^{2k+1} Q(\tau, 1)^{k+1}},$$

from which we deduce the claim by (1.7).

(3) The claim follows directly from (2.2) along with part (1) and (1.7). \square

5.3. Further properties of $\Psi_{-k,D}$ and the proof of Theorem 1.3. We begin with the local behaviour of $\Psi_{-k,D}$. Similar as in the proof of Proposition 4.2, we obtain.

Proposition 5.6. *Let $\tau \in E_D$.*

(1) *We have*

$$\lim_{\varepsilon \rightarrow 0^+} (\Psi_{-k,D}(\tau + i\varepsilon) - \Psi_{-k,D}(\tau - i\varepsilon)) = 0.$$

(2) *We have*

$$\frac{1}{2} \lim_{\varepsilon \rightarrow 0^+} (\Psi_{-k,D}(\tau + i\varepsilon) + \Psi_{-k,D}(\tau - i\varepsilon)) = \Psi_{-k,D}(\tau).$$

(3) *We have*

$$\lim_{\varepsilon \rightarrow 0^+} \left(\frac{\partial}{\partial \bar{\tau}} \Psi_{-k,D}(\tau + i\varepsilon) - \frac{\partial}{\partial \bar{\tau}} \Psi_{-k,D}(\tau - i\varepsilon) \right) = i D^{k+\frac{1}{2}} v^{2k} \sum_{\substack{Q \in \mathcal{Q}_D \\ Q_\tau = 0}} \frac{\operatorname{sgn}(Q)}{Q(\bar{\tau}, 1)^{k+1}}.$$

Furthermore, we have, for every $\tau \in \mathbb{H} \setminus E_D$,

$$\begin{aligned} \xi_{-2k} \left(\Lambda_{k+1,D}^*(\tau) \right) &= \Lambda_{k+1,D}(\tau), & \mathbb{D}^{2k+1} \left(\Lambda_{k+1,D}^*(\tau) \right) &= 0, \\ \xi_{-2k} \left(\mathcal{E}_{\Lambda_{k+1,D}}(\tau) \right) &= 0, & \mathbb{D}^{2k+1} \left(\mathcal{E}_{\Lambda_{k+1,D}}(\tau) \right) &= \Lambda_{k+1,D}(\tau). \end{aligned} \quad (5.3)$$

The third claim follows by holomorphicity of $\mathcal{E}_{\Lambda_{k+1,D}}$, while the second claim holds as $\Lambda_{k+1,D}^*$ (as a function of τ) is a polynomial of degree at most $2k$ by (1.8). The first and fourth claim follow by a standard calculation using the integral representations from (1.8) directly.

Now, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. (2) We define

$$f(\tau) := \Psi_{-k,D}(\tau) + \frac{D^{k+\frac{1}{2}}(2k)!}{(4\pi)^{2k+1}} \mathcal{E}_{\Lambda_{k+1,D}}(\tau) - D^{k+\frac{1}{2}} \Lambda_{k+1,D}^*(\tau).$$

Combining Proposition 5.2 with (5.3), we deduce that

$$\xi_{-2k}(f(\tau)) = \mathbb{D}^{2k+1}(f(\tau)) = 0.$$

Hence, f is a polynomial in τ of degree at most $2k$. By Proposition 5.6 (1), $\Psi_{-k,D}$ has no jumps on E_D . Thus, we may freely select an arbitrary connected component of $\mathbb{H} \setminus E_D$ to compute f . Choosing the connected component of $\mathbb{H} \setminus E_D$ containing $i\infty$, we are in the same situation as in the induction start during the proof of [3, Theorem 7.1]. In other words, the function f is in fact constant, and this constant was computed in [3, Lemma 7.3]. We infer that f coincides with c_∞ .

(1) We verify the four conditions in Definition 2.7.

(i) Modularity of weight $-2k$ follows by Lemma 2.2 and (2.1).

(ii) Local harmonicity with respect to Δ_{-2k} outside E_D is shown in Proposition 5.2 (3).

(iii) The required behaviour on E_D is given in Proposition 5.6 (2).

(iv) By Theorem 1.3 (2), $\Psi_{-k,D}$ has most polynomial growth towards $i\infty$. More precise, $\Lambda_{k+1,D}$ admits a Fourier expansion of the shape $\sum_{n \geq 1} c(n) e^{2\pi i n \tau}$, where the Fourier coefficients depend on the connected component of $\mathbb{H} \setminus E_D$ in which τ is located and which were computed in [23, 24]. Noting that $c(n) \in \mathbb{R}$ and letting $\Gamma(s, y)$ denote the incomplete Γ -function, we obtain, for $v \gg 1$,

$$\mathcal{E}_{\Lambda_{k+1,D}}(\tau) = \sum_{n \geq 1} \frac{c(n)}{n^{2k+1}} e^{2\pi i n \tau}, \quad \Lambda_{k+1,D}^*(\tau) = \sum_{n \geq 1} \frac{c(n)}{(4\pi n)^{2k+1}} \Gamma(2k+1, 4\pi n v) e^{-2\pi i n \tau}.$$

We observe that the holomorphic Eichler integral vanishes as $\tau \rightarrow i\infty$, and the same holds for the non-holomorphic Eichler integral due to [1, §8.11 (i)]. This proves that

$$\lim_{\tau \rightarrow i\infty} \Psi_{-k,D}(\tau) = c_\infty.$$

Proposition 5.6 (1) yields that the singularities of $\Psi_{-k,D}$ on E_D are continuously removable. Combining Proposition 5.6 (3) with Lemmas 2.3 and 4.3 shows that $\Psi_{-k,D}$ has no differentiable continuation to E_D . This completes the proof. \square

REFERENCES

- [1] R. Boisvert, C. Clark, D. Lozier, and F. Olver, *NIST Handbook of Mathematical Functions*, Cambridge University Press (2010).
- [2] K. Bringmann, A. Folsom, K. Ono, and L. Rolén, *Harmonic Maass forms and mock modular forms: theory and applications*, American Mathematical Society Colloquium Publications **64**, American Mathematical Society, Providence, RI, (2017).
- [3] K. Bringmann, B. Kane and W. Kohnen, *Locally harmonic Maass forms and the kernel of the Shintani lift*, Int. Math. Res. Not. IMRN (2015), no. 11, 3185–3224.
- [4] K. Bringmann, A. Mono and L. Rolén, *Flipping operators and locally harmonic Maass forms*, Ramanujan J. **68** (2025), no. 2, 40.
- [5] J. Bruinier and J. Funke, *On two geometric theta lifts*, Duke Math. J. **125** (2004), 45–90.
- [6] J. Bruinier, K. Ono, and R. Rhoades, *Differential operators for harmonic weak Maass forms and the vanishing of Hecke eigenvalues*, Math. Ann. **342** (2008), 673–693.
- [7] W. Duke, O. Imamoglu, and A. Tóth, *Rational period functions and cycle integrals*, Abh. Math. Semin. Univ. Hambg. **80** (2010), no. 2, 255–264.

- [8] W. Duke, O. Imamoglu, and A. Tóth, *Cycle integrals of the j -function and mock modular forms*, Ann. of Math. (2) **173** (2011), 947–981.
- [9] M. Eichler, *Eine Verallgemeinerung der Abelschen Integrale*, Math. Z. **67** (1957), 267–298.
- [10] M. Hövel, *Automorphe Formen mit Singularitäten auf dem hyperbolischen Raum*, Ph.D. Thesis, TU Darmstadt, (2012).
- [11] Ö. Imamoglu and C. O’Sullivan, *Parabolic, hyperbolic and elliptic Poincaré series*, Acta Arith. **139** (2009), no. 3, 199–228.
- [12] S. Katok, *Closed geodesics, periods and arithmetic of modular forms*, Invent. Math. **80** (1985), no. 3, 469–480.
- [13] M. Knopp, *Modular integrals and their Mellin transforms*, (1990), 327–342, *Analytic number theory*, Allerton Park, IL, (1989), Progr. Math. **85**, Birkhäuser Boston, Boston, MA.
- [14] M. Knopp, *On abelian integrals of the second kind and modular functions*, Amer. J. Math. **84** (1962), 615–628.
- [15] M. Knopp, *On generalized abelian integrals of the second kind and modular forms of dimension zero*, Amer. J. Math. **86** (1964), 430–440.
- [16] M. Knopp, *Some new results on the Eichler cohomology of automorphic forms*, Bull. Amer. Math. Soc. **80** (1974), 607–632.
- [17] M. Knopp, *Rational period functions of the modular group*, Duke Math. J. **45** (1978), 47–62.
- [18] M. Knopp, *Rational period functions of the modular group. II*, Glasgow Math. J. **22** (1981), no. 2, 185–197.
- [19] W. Kohnen, *Fourier coefficients of modular forms of half-integral weight*, Math. Ann. **271** (1985), 237–268.
- [20] W. Kohnen and D. Zagier, *Values of L -series of modular forms at the center of the critical strip*, Invent. Math. **64** (1981), 175–198.
- [21] W. Kohnen and D. Zagier, *Modular forms with rational periods*, *Modular forms*, Durham, (1983) Ellis Horwood Ser. Math. Appl.: Statist. Oper. Res., Horwood, Chichester, (1984), 197–249.
- [22] J. Lehner, *The Eichler cohomology of a Kleinian group*, Math. Ann. **192** (1971), 125–143.
- [23] A. Mono, *Eisenstein series of even weight $k \geq 2$ and integral binary quadratic forms*, Proc. Amer. Math. Soc. **150** (2022), no. 5, 1889–1902.
- [24] A. Mono, *Locally harmonic Maass forms of positive even weight*, Israel J. Math. **261** (2024), no. 2, 671–694.
- [25] L. Parson, *Modular integrals and indefinite binary quadratic forms*, *A tribute to Emil Grosswald: number theory and related analysis*, Contemp. Math. **143**, Amer. Math. Soc., Providence, RI (1993), 513–523.
- [26] G. Shimura, *On modular forms of half integral weight*, Ann. of Math. (2) **97** (1973), 440–481.
- [27] T. Shintani, *On construction of holomorphic cusp forms of half integral weight*, Nagoya Math. J. **58** (1975), 83–126.
- [28] J. Stienstra and D. Zagier, *Bimodular forms and holomorphic anomaly equation*, In: Workshop on Modular Forms and String Duality, Banff International Research Station, 2006, <https://www.birs.ca/workshops/2006/06w5041/report06w5041.pdf>
- [29] D. Zagier, *Modular forms associated to real quadratic fields*, Invent. Math. **30** (1975), 1–46.
- [30] D. Zagier, *Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields*, *Modular functions of one variable, VI*, Proc. Second Internat. Conf., Univ. Bonn, Bonn (1976), Springer, Berlin, (1977) 105–169. Lecture Notes in Math., Vol. 627.
- [31] D. Zagier, *Zetafunktionen und quadratische Körper*, Hochschultext, Springer-Verlag, Berlin, 1981.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, DIVISION OF MATHEMATICS, UNIVERSITY OF COLOGNE, WEYERTAL 86–90, 50931 COLOGNE, GERMANY
 Email address: kbringma@math.uni-koeln.de

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, DIVISION OF MATHEMATICS, UNIVERSITY OF COLOGNE, WEYERTAL 86–90, 50931 COLOGNE, GERMANY
 Current address: DEPARTMENT OF MATHEMATICS, VANDERBILT UNIVERSITY, 1326 STEVENSON CENTER, NASHVILLE, TN 37240, USA
 Email address: andreas.mono@vanderbilt.edu